Supersymmetric non-linear unification in particle physics. Kahler manifolds, bundles for matter representations and anomaly cancellation

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Supersymmetric non-linear unification in particle physics

Kähler manifolds, bundles for matter representations and anomaly cancellation

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Contents

Chapter 1

Introduction

The search for a fundamental theory of nature is one of the tasks that physicists have set themselves. Within such a framework all phenomena in nature can in principle be understood. Even if we are able to find such a theory in the near future it would not imply at all that physics is finished. As this theory describes physics on the smallest scales, it will probably be too complicated to work out the details of all kinds of chemical and biological processes. For example, the knowledge of the interactions of the building blocks of protons, called quarks and gluons, does not help much in understanding how the collection of brain-cells enables us to think about such questions.

This thesis constitutes just a modest step in the realization of the dream of a complete theory of all physical phenomena in nature. Before we can explain the details of the research discussed in this thesis, we first give an overview of related developments in elementary particle physics by briefly describing the standard model, unification theory, supersymmetry and string theory. After this tour through the world of particle physics we are able to explain the motivation for and the ingredients of the work described in the remainder of this thesis.

We now review the standard model of particle physics: the theory that describes the interaction between all the elementary particles known to us. We refer to the standard literature [1, 2, 3, 4, 5, 6, 7, 8] for a more complete treatment and the necessary field theory background. There are four known fundamental interactions. The most familiar force is, of course, gravity, that causes the apple to fall to the ground and at the same time keeps communication satellites in orbit around the earth. The electro-magnetic force binds the atoms and molecules in our body and allows us to use these satellites for communication. The other two fundamental forces only act on subatomic scales. The so-called strong force provides the glue between protons and neutrons that is necessary to form the nuclei of all the atoms we know. A lot of nuclei are unstable; they can decay under the influence of the weak force. The mathematical characterization of these different interactions is achieved by group theory. The gauge groups of the electro-magnetic, strong and weak interactions are the Abelian $U_{em}(1)$ and the non-Abelian $SU(3)$ and $SU_L(2) \times U_Y(1)$ groups, respectively. The electro-magnetic gauge group $U_{em}(1)$ is a subgroup of the electro-weak group $SU_L(2) \times U_Y(1)$, hence there is a "unified" description of weak and electro-magnetic interactions possible. The number of generators of a group determines the number of different gauge fields: $U_{em}(1)$ has just one generator while $SU(3)$ has eight, which explains why there is just one photon but eight gluons. Apart from the electric charge generator, $SU_L(2) \times U_Y(1)$ contains three more generators, hence there are three more gauge bosons: W^+, W^- and Z. They turn out to be massive. Gravitons, the hypothetical messenger particles of gravity, can be interpreted as the gauge bosons of the Poincaré group in a more or less similar fashion.

The fundamental building blocks of nature are the quarks and leptons. These fermions interact with each other via the forces of nature discussed above. The strong interaction confines the quarks to reside inside protons and neutrons only; it is impossible to produce a free quark in an experiment. The leptons, e.g. the electron and the neutrino, do not feel the strong interactions. Large accelerator experiments have shown that there are three families or generations of quarks and leptons. Unlike the strong and electro-magnetic interactions the weak interaction ($SU_L(2)$) only couples to left-handed fermions[1]. As the standard model treats left- and right-handed particles of the same type differently it is called a chiral theory.

The interactions of the particles of the different families with the forces of the standard model are identical; only their masses differ. However because of their chiral nature, the fermions should all be massless. This problem is overcome by the introduction of a $SU_L(2)$-Higgs doublet of complex scalar particles. The so-called Yukawa interactions between this Higgs doublet and the fermions result in effective masses for the fermions. This happens if the real neutral part of the Higgs doublet acquires a non-vanishing vacuum expectation value. This is called spontaneous symmetry breaking as the theory remains invariant under the electro-weak symmetries, but the vacuum state breaks most of them. The $W^{\pm}$ and the Z bosons also become massive because of the vacuum expectation value of the Higgs; their longitudinal components come from the three Goldstone bosons that correspond to the broken symmetries. This implies that of the 4 real states of the Higgs doublet just one physical particle survives. This Higgs-boson has not yet been observed.

Even though the standard model has been extremely successful in describing the phenomena of high-energy experiments [9], theorists feel that it cannot be the final answer. We now discuss and motivate various extensions of the standard

[1] Left- and right-handed fermions have positive and negative eigenvalues, respectively, under the action of the chirality operator γ_5. For (almost) massless particles this is the same as the projection of the spin on the direction of motion (helicity) of the fermion.

model, starting with grand unification.

The standard model is rather complicated because it contains many arbitrary choices. For example, why is nature described by the gauge group $SU(3) \times SU_L(2) \times U_Y(1)$? Another curious thing of the standard model is that the gauge coupling constants evolve with the energy scale such that they come quite close together at the so-called unification scale of about 10^{15} GeV. Is this merely a coincidence?

A more technical thing that seems to happen by accident is that the standard model is free of gauge anomalies. A symmetry of the classical theory is said to be anomalous if it is not a symmetry of the quantum theory. As anomalies are true quantum effects, let us explain what the conflict is. The quantum version of a field theory is often plagued with infinities due to large momenta. To make sense out of such a theory one has to control these infinities somehow by regulating them. However, not always does a regulator exist that respects the whole structure of the classical theory. The famous dimensional regularization [10] cannot extend all properties of the chirality operator γ_5 to arbitary (complex) dimensions. This implies that all chiral theories run a risk of developing anomalies. An anomalous global symmetry is simply bad luck: the theory is just more complicated than one would expect from the classical description. The decay of the π_0 into two photons is the most familiar example of this situation. An anomalous gauge symmetry has disastrous consequences as it makes the theory inconsistent: the photon acquires a time-like polarization. Even though the standard model is a chiral theory it contains no gauge anomalies. For some mysterious reason the quantum numbers of the chiral fermions cause all dangerous anomalies to cancel.

These remarkable properties can be explained within the framework a unification theory [11, 12]. In such a theory the gauge group of the standard model is enlarged to a single simple Lie-group; hence it describes gauge interactions with just one coupling constant. However, as we do not observe the consequences of this larger group directly, the unification group has to be broken to the standard model group at the unification scale 10^{15} GeV. Below this threshold the additional gauge bosons and scalar Higgs-bosons should become heavy due to an effect similar to the standard model Higgs mechanism. As the large symmetry group is broken below this scale, we recover the standard model coupling constants but they now unify by construction. The smallest group one can use for this purpose is $SU(5)$. However in its simplest form it has some phenomenological problems, like proton instability, and it fails to explain why the standard model turns out to be anomaly-free. The gauge groups $SO(10)$ and E_6 are more promising as they have only anomaly-free representations and leave enough room to make the proton lifetime longer than the observed bounds.

Supersymmetry [13, 12, 14] is the next extension of the standard model we describe. In the standard model the fermions and bosons have completely different properties: they are contained in different representations of the gauge groups. It is possible to construct a theory in which the bosons and fermions have the same standard model quantum numbers except, of course, for their spin. The resulting theory possesses an additional symmetry between fermion and bosons called supersymmetry. The bosons and fermions that transform into each other under supersymmetry form a so-called supersymmetric multiplet, or multiplet for short. In the following we encounter three different types of multiplets: the chiral, the vector and the supergravity multiplets. A chiral (or scalar) multiplet consists of a complex scalar field and a chiral fermion. A vector multiplet contains a gauge field and a Majorana fermion. And finally the supergravity multiplet consists of the graviton and its superpartner the gravitino. One motivation for a supersymmetric extension of the standard model is, that it can explain why the mass of the Higgs boson (≤ 500 GeV) is not of the same order as the unification scale or the Planck scale 10^{19} GeV. Only by extremely tedious fine-tuning at every order of perturbation theory can the large difference between the electro-weak scale of about 100 GeV and the unification or the Planck scale be maintained within the standard model . This strong numerical dependence on large mass scales within the standard model is due to quadratic divergences in the scalar Higgs sector. In a supersymmetric theory these quadratic divergences are absent because bosonic and fermionic loop contributions cancel each other.

One of the curious consequences of supersymmetry is that the unification of the coupling constants works better [15]: in the standard model they come close to each other at the unification scale, whereas in the supersymmetric case they become really identical within the measuring accuracy. This indicates that it may be profitable to combine supersymmetry and grand unification [16].

The Poincaré group can be extended in an elegant fashion to the super-Poincaré group. A consequence of this is that if supersymmetry is gauged, the theory automatically also includes gravity. As the quantum theory of supergravity is better behaved than that of gravity alone, supergravity might be a necessary ingredient of a theory of quantum gravity.

The theories described above contain a lot of appealing ideas but all of them have to be put in by hand. Over the last fifteen years there have been extensive studies of a theory where many of the ingredients discussed above come out automatically: string theory [17, 18, 19, 20, 21, 22]. In string theory the fundamental objects are no longer point particles like in quantum field theory, but one dimensional strings.[2] The different modes of vibration of a string correspond to different particles in a field theory description. One of the vibrations which is always present in the string spectrum is the graviton, hence string theory always contains gravity. Only five consistent string theories have been constructed so far, all of which are supersymmetric and live in 10 dimensions. Therefore string theory cannot be used directly to describe the real world. By compactification, 6

[2] String theories also include higher dimensional objects, like membranes.

of the 10 dimensions are curled up to a very small size, so that these additional dimensions are not accessible for us today.[3] Unfortunately, there is no unique way to perform this compactification [23, 24, 25]. The phenomenologically most promising string theory, the heterotic $E_8 \times E_8$ string theory, can produce a multitude of standard-like models [26].

We are now in a position to describe the main ideas of this thesis. We study globally and locally supersymmetric models based on Kähler manifolds. Before we explain in more detail what this means, we should first get an intuitive picture of the situation under investigation. A manifold is a smooth surface of arbitrary dimension. The geometry of a manifold is non-trivial if it is curved. Consider a simple non-trivial manifold: the sphere. It turns out that it is possible to associate a supersymmetric field theory with the sphere: the scalar fields parameterize the sphere. In section 2.2.1 we explain how this works in some detail. The Higgs mechanism in the standard model provides the reader with a particle physics background with another example of a field theory based on a non-trivial manifold. The scalar potential there has a shape similar to the Mexican hat potential. The (real) neutral scalar component of the Higgs-doublet acquires a non-zero vacuum expectation value. However, this choice is far from unique. In general there exist a whole manifold of field configurations that minimize the potential. The Goldstone bosons parameterize this so-called coset space manifold. The broken group elements determine how one can move through this vacuum manifold. They turn out to be represented by non-linear transformations of the Goldstone bosons. This means that the action of the broken symmetries cannot be written as matrix multiplication of the scalars.

If a supersymmetric model contains non-linear symmetries, this has non-trivial geometrical consequences, as we saw above. In general, supersymmetry requires the manifold of complex scalars of the chiral multiplets to be of a special type, called a Kähler manifold. The kinetic terms of the chiral multiplets are described by the metric of this Kähler manifold. Therefore the non-linear symmetries of the model should have the geometrical interpretation of isometries, symmetries of the metric. The resulting model is of the so-called supersymmetric non-linear sigma-model (σ-model) type, as it contains scalars living on a manifold with non-trivial geometry. (The notion of a sigma-model originates from a linear and non-linear description of pions; in the linear description there is an additional partner of the pions called the sigma-field.) The non-linearity of the model will introduce some sort of mass parameter to give all terms in the Lagrangean the right mass dimensions; we call this the sigma-model mass scale. This scale measures in some sense the non-linearity of the model.

Non-linear symmetries also play an important role in compactifications of

[3] As string theory contains higher dimensional objects, another possibility is that we live on a (mem)brane with three space dimensions.

string theory. The moduli space, the space of different string compactification configurations, is described by a compact Kähler manifold. If the compactification manifold is a torus or an orbifold (a torus with some points identified), the moduli space takes the form of a coset space. In this thesis we assume that a supersymmetric model contains non-linear symmetries, leaving their origin unspecified.

Before we study the phenomenology of a supersymmetric model based on a non-trivial manifold, we first have to convince ourselves that the theory makes sense at all. In particular it should not contain any dangerous anomalies like the ones that gauge theories can be plagued with. For a general introduction to various aspects of anomalies the reader might want to consult [2]. If there are anomalies in local symmetries and in the isometries of the model one has to find a way to remove them. One way to cancel the anomalies is by introducing Wess-Zumino counter terms [27, 28]. Another option is to introduce additional chiral multiplets that cancel the anomalies. It is this latter route of matter coupling that we consider in this thesis. However, matter should be added in such a way that all isometries of the original model are respected, else the isometries cannot be promoted to become symmetries of the full model.

Matter-coupling to a supersymmetric σ-model is not a trivial step: the additional chiral multiplets should form representations of the symmetries of the original model and should be well defined over the whole original Kähler manifold. The latter does not happen automatically for a topologically non-trivial manifold. Several coordinate systems are needed to describe such a manifold. We need at least the northern and southern hemispheres to describe the sphere. One has to make sure that the descriptions of the matter fields on the different coordinate patches are compatible. The latter is crucial because the enlarged supersymmetric model should again be based on a Kähler manifold. To achieve the anomaly cancellation it is necessary to be able to construct various representations of the symmetries of the original model. The simplest example of a supersymmetric matter representation is provided by a chiral multiplet that transforms as a tangent vector of the manifold. More complicated matter representations can be obtained by considering tensor products of this structure. These representations alone are often not sufficient to obtain anomaly-free models in a non-trivial way. However, it turns out to be also possible to obtain a non-trivial singlet chiral superfield matter representation using the transformation properties of the Kähler potential. This singlet chiral multiplet has to satisfy a consistency requirement, called the cocycle condition of the line bundle, which leads to quantization of its charge in many cases. This charge quantization may be in conflict with the charges needed for anomaly cancellation. Therefore it is non-trivial to see whether consistent matter coupling is possible in such a way that anomalies cancel.

In this thesis we focus in particular on models that are based on the geometry of Kähler coset spaces G/H, where H is a subgroup of a group G. These models provide us with a large class of examples of non-trivial manifolds, for which we can do more concrete and explicit calculations. An extended discussion of various aspects of such non-linear realizations can be found in ref. [29]. Some of these cosets might well have phenomenological interest [30, 31, 32], especially when they contain a unification group like the cosets $E_6/[SO(10)\times U(1)]$, $SO(10)/U(5)$ and $SU(5)/[SU(2)\times U(1)\times SU(3)]$. According to refs. [33, 34], supersymmetric models based on this type of coset space G/H are claimed inconsistent because of anomalies and not interesting for phenomenology because they cannot arise from symmetry breaking. We do not completely agree with these conclusions: although a supersymmetric model based on a pure Kähler coset contains anomalies, they can often be cancelled by additional matter representations. As for the second conclusion, we treat the models in their own right and do not speculate on their origin.

In order to describe various aspects of supersymmetric models based on coset spaces, we discuss how their non-linear transformations can be obtained and described, considering both infinitesimal and finite transformations. We review the general machinery to construct Kähler potentials, from which the metric of the coset can be derived, and develop some additional techniques. Matter coupling to coset spaces has some special additional properties, which allow for more different types of matter representations. We give explicit representations of non-trivial singlet matter representations and determine the smallest charge that a non-trivial singlet chiral multiplet can carry.

We show that consistent supersymmetric models based on coset spaces exist by constructing explicit examples. These examples are $E_6/[SO(10)\times U(1)]$, $SU(5)/[SU(2)\times U(1)\times SU(3)]$ and $SO(10)/U(5)$. For these cosets we have made a first step in the analysis of their phenomenology within the context of global or local supersymmetry. More detailed analyses will have to reveal whether or not these types of non-linear sigma-models are relevant for phenomenology.

We conclude the introduction with an outline of the remainder of this thesis. We review the general features of supersymmetric models based on Kähler manifolds in chapter 2. In particular we discuss anomalies of supersymmetric sigma-models. Chapter 3 is devoted to matter coupling to supersymmetric models. It turns out that for consistency matter representations have to be interpreted as sections of bundles. We discuss the coupling of chiral multiplets to supergravity in the same language. A technical discussion on the construction of block-diagonal metrics for σ-model/matter systems concludes this chapter. In chapter 4 we construct Kähler potentials and matter representations for Kähler coset spaces. As chapters 2 to 4 are quite general, we illustrate various aspects by studying supersymmetric anomaly-free models based on a sphere and a hyperbolic space. Chapters 5 to 7 provide us with more complicated examples of such models based on coset spaces. Chapter 5 discusses an extension of the standard model involving non-linear $SU(5)$ symmetries based on the coset $SU(5)/[SU(2) \times U(1) \times SU(3)]$. Chapters 6 and 7 are devoted to the construction of anomaly-free models based on the cosets $SO(2N)/U(N)$ and $E_6/[SO(10) \times U(1)]$, respectively. Some phenomenological aspects of these models are analyzed. This investigation is performed in the context of supergravity in chapter 5, while we restrict ourselves to global supersymmetry in chapters 6 and 7. Chapter 8 contains the conclusions. Various more mathematical aspects are discussed in the appendices.

Chapter 2

Kähler Manifolds in Supersymmetric Models

2.1 Introduction

Supersymmetry imposes many restrictions on a theory; if it contains a collection of chiral multiplets, supersymmetry forces the local geometry of their scalars to be of a special type: they span a so-called Kähler manifold [35]. This is a complex manifold with the additional property that all its geometrical properties are encoded in one real scalar function, called the Kähler potential. Section 2.2 describes general geometrical features of Kähler manifolds using the Kähler potential, the isometries and the vielbein formalism. The emphasis of our discussion will be on the applications to supersymmetric model building rather than mathematical rigor; for this the reader may consult appendix A. Some aspects of the general discussion are illustrated with two simple examples of Kähler manifolds: the 2-dimensional sphere and the 2-dimensional hyperbolic space in 2.2.1.

The construction of globally supersymmetric Lagrangeans [13, 14, 36] is discussed in section 2.3. The kinetic part of a Lagrangean for chiral multiplets is determined by the metric of the Kähler manifold. The Yukawa-terms in a supersymmetric Lagrangean arise from a holomorphic function called the superpotential. Another consequence of supersymmetry is that the kinetic terms of the vector multiplets are determined by a holomorphic function of the chiral multiplets. The real part of this function can be interpreted as the gauge coupling constant. The emphasis is on the incorporation of non-linear symmetries and their consequences. The section is concluded with a discussion of two non-linear generalizations of the Wess-Zumino model, based on the two-sphere and the hyperbolic space.

The classical description of a supersymmetric model of section 2.3 gives a rough idea of the properties of the model at the quantum level as well. However the chiral fermions within the chiral multiplets can lead to anomalous symmetries: classical symmetries that are broken in the quantum theory. Section 2.4 discusses the calculation and consequences of anomalies in supersymmetric models. The strategy of the remainder of this thesis is put forward: anomalies in supersymmetric non-linear σ-models can be cancelled by including additional chiral fermions contained in chiral multiplets. These so-called matter multiplets are the main topic of chapter 3.

2.2 Kähler Manifolds

In this section we discuss some elements of Kähler geometry for physical applications, the more precise mathematical issues can be found in appendix A or in refs. [37, 38, 39, 18]. We first give the definitions of the metric, Kähler-form, connection and curvature on a Kähler manifold. Next we discuss the issue of the global definition of a manifold. The isometries of a Kähler manifold are discussed in the language of Killing vectors and potentials. After that the transformation of (co)tangent vectors and the vielbein formalism are reviewed. The section is concluded with a discussion of two Kähler manifolds: the sphere and the hyperbolic space.

A complex manifold is a manifold on which globally the notion of square root of $-\mathbb{1}$ in the tangent space can be defined. Locally such a manifold can be described by complex coordinates $Z^{\mathcal{A}}$. A function of $Z^{\mathcal{A}}$ only is called holomorphic or analytic; a function of the complex conjugate of these coordinates $\bar{Z}^{\underline{\mathcal{A}}}$ is called anti-holomorphic. A Kähler manifold is a complex manifold that has a complex Hermitean metric $G_{\underline{\mathcal{A}}\mathcal{A}}$ that locally can be expressed as the second mixed derivative

$$G_{\underline{\mathcal{A}}\mathcal{A}} = \mathcal{K}_{,\underline{\mathcal{A}}\mathcal{A}} \tag{2.1}$$

of a real function $\mathcal{K}(\bar{Z}, Z)$, called the Kähler potential. Here the comma denotes differentiation with respect to $\bar{Z}^{\underline{\mathcal{A}}}$ and $Z^{\mathcal{A}}$. The indices $\mathcal{A}$ and $\underline{\mathcal{A}}$ are independent and summation over repeated indices is understood throughout this thesis. The complex connections of a Kähler manifold are given by

$$\Gamma^{\mathcal{A}}_{\mathcal{B}\mathcal{C}} = G^{\mathcal{A}\underline{\mathcal{A}}} G_{\underline{\mathcal{A}}\mathcal{B},\mathcal{C}}, \quad \bar{\Gamma}^{\underline{\mathcal{A}}}_{\underline{\mathcal{B}}\underline{\mathcal{C}}} = G^{\mathcal{A}\underline{\mathcal{A}}} G_{\underline{\mathcal{B}}\mathcal{A},\underline{\mathcal{C}}}, \tag{2.2}$$

where $G^{\mathcal{A}\underline{\mathcal{A}}}$ denotes the inverse of the metric $G_{\underline{\mathcal{A}}\mathcal{A}}$. The connections with both holomorphic ($\mathcal{A}$) and anti-holomorphic ($\underline{\mathcal{A}}$) indices vanish. The curvature tensor also takes a very special form

$$R_{\underline{\mathcal{A}}\mathcal{A}\underline{\mathcal{B}}\mathcal{B}} = G_{\underline{\mathcal{A}}\mathcal{A},\underline{\mathcal{B}}\mathcal{B}} - G_{\underline{\mathcal{A}}\mathcal{C},\underline{\mathcal{B}}} G^{\mathcal{C}\underline{\mathcal{C}}} G_{\underline{\mathcal{C}}\mathcal{A},\mathcal{B}}. \tag{2.3}$$

Using these expressions it is easy to show that

$$\Gamma^{\mathcal{B}}_{\mathcal{A}\mathcal{B}} = G^{\mathcal{B}\underline{\mathcal{B}}} G_{\underline{\mathcal{B}}\mathcal{A},\mathcal{B}} = (\ln\det G)_{,\mathcal{A}}, \qquad R_{\underline{\mathcal{A}}\mathcal{A}} = \Gamma^{\mathcal{B}}_{\mathcal{A}\mathcal{B},\underline{\mathcal{A}}} = (\ln\det G)_{,\underline{\mathcal{A}}\mathcal{A}}. \tag{2.4}$$

An alternative definition of a Kähler manifold is given by the demand that the Kähler form $\omega(\mathcal{K})$

$$\omega(\mathcal{K}) = -iG_{\underline{\mathcal{A}}\mathcal{A}} d\bar{Z}^{\underline{\mathcal{A}}} \wedge dZ^{\mathcal{A}} \tag{2.5}$$

is closed: $d\omega = 0$. For the precise definition of forms, the wedge product $\wedge$ and the exterior derivate d, we refer to appendix A. The constraint that $d\omega = 0$ can be written in terms of the metric as

$$G_{\underline{\mathcal{A}}\mathcal{A},\mathcal{B}} = G_{\underline{\mathcal{A}}\mathcal{B},\mathcal{A}}, \quad G_{\underline{\mathcal{A}}\mathcal{B},\underline{\mathcal{B}}} = G_{\underline{\mathcal{B}}\mathcal{A},\underline{\mathcal{A}}}. \tag{2.6}$$

It can be shown that these conditions are equivalent to having a metric $G_{\underline{\mathcal{A}}\mathcal{A}}$ determined by a Kähler potential $\mathcal{K}$, see (2.1). It should be noted that changing the Kähler potential

$$\mathcal{K}(\bar{Z}, Z) \longrightarrow \mathcal{K}'(\bar{Z}, Z) = \mathcal{K}(\bar{Z}, Z) + \mathcal{F}(Z) + \bar{\mathcal{F}}(\bar{Z}) \tag{2.7}$$

with a sum of a holomorphic $\mathcal{F}$ and anti-holomorphic $\bar{\mathcal{F}}$ function leads to the same metric and hence the same geometry. This transformation is called a Kähler transformation.

When a manifold $\mathfrak{M}$ can be covered by one complex coordinate system, the definitions above are sufficient. There are a lot of complex manifolds that cannot be described by just one coordinate system: several coordinate patches $U_{(a)}$, enumerated by a, have to be introduced on the manifold. The different coordinate patches are glued together on their overlaps: a point p in an overlap $U_{(a)} \cap U_{(b)}$ is described by two sets of coordinates $Z_{(a)}$ and $Z_{(b)}$. A function on the manifold has to respect this gluing. These points are discussed in more mathematical detail in the appendix A.

An infinitesimal isometry $\delta_i Z^{\mathcal{A}}$ of the metric

$$\delta_i Z^{\mathcal{A}} = \mathcal{R}_i^{\mathcal{A}}(Z) \tag{2.8}$$

is generated by a Killing vector $\mathcal{R}_i^{\mathcal{A}}(Z)$, that satisfies the Killing conditions

$$(G_{\underline{\mathcal{A}}\mathcal{B}} \mathcal{R}_i^{\mathcal{B}})_{,\mathcal{A}} + (\bar{\mathcal{R}}_i^{\underline{\mathcal{B}}} G_{\underline{\mathcal{B}}\mathcal{A}})_{,\underline{\mathcal{A}}} = 0. \tag{2.9}$$

Denote by $\{\mathcal{R}_i^{\mathcal{A}}(Z)\}$ a complete set of Killing vectors that are enumerated by $i, j, \ldots$. Since the Lie-derivative $\mathcal{R}_{[i}^{\mathcal{B}} \mathcal{R}_{j],\mathcal{B}}^{\mathcal{A}} = \mathcal{R}_i^{\mathcal{B}} \mathcal{R}_{j,\mathcal{B}}^{\mathcal{A}} - \mathcal{R}_j^{\mathcal{B}} \mathcal{R}_{i,\mathcal{B}}^{\mathcal{A}}$ satisfies the Killing conditions as well, it follows that

$$\mathcal{R}_{[i}^{\mathcal{B}} \mathcal{R}_{j],\mathcal{B}}^{\mathcal{A}} = f_{ij}{}^k \mathcal{R}_k^{\mathcal{A}}, \tag{2.10}$$

where the structure coefficients $f_{ij}{}^k$ of the algebra of the isometry group G are introduced. Here $[i \ldots j]$ denotes anti-symmetrization over the indices inside, without a symmetrization factor. The Lie group G is generated by the algebra with the basis $\{T_i\}$, satisfying the commutation relations

$$[T_i, T_j] = f_{ij}{}^k T_k. \tag{2.11}$$

The Jacobi identities

$$[T_i, [T_j, T_k]] - [T_j, [T_i, T_k]] = [[T_i, T_j], T_k], \tag{2.12}$$

guarantee that $[T_i, .]$ defines a representation of the algebra. This imposes constraints on the structure coefficients

$$f_{il}{}^m f_{jk}{}^l + f_{jl}{}^m f_{ki}{}^l + f_{kl}{}^m f_{ij}{}^l = 0. \tag{2.13}$$

The Killing metric η_{ij} is defined from the structure coefficients

$$-2\eta_{ij} = f_{ik}{}^l f_{jl}{}^k. \tag{2.14}$$

More details on Lie groups in general can be found in refs. [40, 41, 42, 43] and their applications to (particle) physics in refs. [11, 44, 45]

The Kähler potential $\mathcal{K}$ does not have to be invariant but may transform under the isometries (2.8) as

$$\delta_i \mathcal{K}(\bar{Z}, Z) = \mathcal{K}_{,\mathcal{A}} \mathcal{R}_i^{\mathcal{A}} + \mathcal{K}_{,\underline{\mathcal{A}}} \bar{\mathcal{R}}_i^{\underline{\mathcal{A}}} = \mathcal{F}_i(Z) + \bar{\mathcal{F}}_i(\bar{Z}), \tag{2.15}$$

where the functions $\mathcal{F}_i$ ($\bar{\mathcal{F}}_i$) are (anti-)holomorphic, which is a specific example of the Kähler transformation (2.7). It should be noticed that $\mathcal{F}_i$ cannot be equal to $\mathcal{K}_{,\mathcal{A}} \mathcal{R}_i^{\mathcal{A}}$ as the former is defined to be holomorphic while the latter clearly is not. By using the group property of the isometries $\delta_{[i}\mathcal{F}_{j]} + \delta_{[i}\bar{\mathcal{F}}_{j]} = \delta_{[i}\delta_{j]}\mathcal{K} = f_{ij}{}^k \delta_k \mathcal{K} = f_{ij}{}^k(\mathcal{F}_k + \bar{\mathcal{F}}_k)$ and the fact that $\mathcal{F}_i$ and $\mathcal{R}_i$ are both holomorphic, it follows that $\delta_{[i}\mathcal{F}_{j]}$ is determined by the structure constants, up to an imaginary constant part: $\delta_{[i}\mathcal{F}_{j]} = f_{ij}{}^k \mathcal{F}_k + i a_{ij}$. Here a_{ij} are real and anti-symmetric constants because the only purely imaginary holomorphic function is the constant function. By an appropriate shift of the functions $\mathcal{F}_i$ these constants can be absorbed into the definition of $\mathcal{F}_i$, which gives

$$\delta_{[i}\mathcal{F}_{j]} = f_{ij}{}^k \mathcal{F}_k. \tag{2.16}$$

These holomorphic functions play a very important role in this thesis. In general it is not easy to give a more explicit formula from which these holomorphic functions $\mathcal{F}_i$ can be derived. However in the case of an Einstein space for which the Ricci tensor $R_{\underline{\mathcal{A}}\mathcal{A}}$ is proportional to the metric: $R_{\underline{\mathcal{A}}\mathcal{A}} = f^2 G_{\underline{\mathcal{A}}\mathcal{A}}$, with proportionality factor f^2, we find that the holomorphic functions are in turn given by $\mathcal{F}_i = \frac{1}{2} f^{-2}\, \mathcal{R}_{i,\mathcal{A}}^{\mathcal{A}}$. This follows from eq. (2.4). The real Killing potentials $\mathcal{M}_i(\bar{Z}, Z)$ are defined by

$$-i\mathcal{M}_i \equiv \mathcal{K}_{,\mathcal{A}} \mathcal{R}_i^{\mathcal{A}} - \mathcal{F}_i = -\mathcal{K}_{,\underline{\mathcal{A}}} \bar{\mathcal{R}}_i^{\underline{\mathcal{A}}} + \bar{\mathcal{F}}_i, \tag{2.17}$$

with the second identity following from eq. (2.15). The Killing vectors $\mathcal{R}_i^{\mathcal{A}}$ can then be obtained from them by differentiation of the Killing potentials $\mathcal{M}_i$:

$$-i\mathcal{M}_{i,\underline{\mathcal{A}}} = G_{\underline{\mathcal{A}}\mathcal{A}}\mathcal{R}_i^{\mathcal{A}}, \qquad -i\mathcal{M}_{i,\mathcal{A}} = -G_{\underline{\mathcal{A}}\mathcal{A}}\bar{\mathcal{R}}_i^{\underline{\mathcal{A}}}. \tag{2.18}$$

These expressions clearly satisfy the Killing conditions (2.9). The Killing potentials can be defined so as to transform under the isometries in the adjoint representation

$$\delta_i\mathcal{M}_j = f_{ij}{}^k\mathcal{M}_k. \tag{2.19}$$

To show this, first observe that $\delta_i\mathcal{M}_j = iG_{\underline{\mathcal{A}}\mathcal{A}}\bar{\mathcal{R}}_{[i}^{\underline{\mathcal{A}}}\mathcal{R}_{j]}^{\mathcal{A}}$ is anti-symmetric. Then

$$\delta_i\mathcal{M}_j = \frac{1}{2}\delta_{[i}\mathcal{M}_{j]} = \frac{i}{2}\left(\mathcal{K}_{,\mathcal{A}}\mathcal{R}_{[j,\mathcal{B}}^{\mathcal{A}}\mathcal{R}_{i]}^{\mathcal{B}} - \delta_{[i}\mathcal{F}_{j]} + G_{\underline{\mathcal{A}}\mathcal{A}}\bar{\mathcal{R}}_{[i}^{\underline{\mathcal{A}}}\mathcal{R}_{j]}^{\mathcal{A}}\right), \tag{2.20}$$

from which the transformation rule for $\mathcal{M}_j$ follows by eqs. (2.10) and (2.16).

We now turn to the transformation properties of (co)tangent vector fields. For a mathematically more rigorous introduction the reader is referred to appendix A. Using the metric of a Kähler manifold, an invariant-line-element is defined by $ds^2 = G_{\underline{\mathcal{A}}\mathcal{A}}d\bar{Z}^{\underline{\mathcal{A}}}dZ^{\mathcal{A}}$ on it. This introduces the notion of infinitesimal distance on the manifold. Under a change of coordinates $Z \longrightarrow Z'(Z)$ the differentials and derivatives transform as

$$dZ^{\mathcal{A}} \longrightarrow dZ'^{\mathcal{A}} = X^{\mathcal{A}}_{\ \mathcal{B}}dZ^{\mathcal{B}}, \qquad \frac{\partial}{\partial Z^{\mathcal{A}}} \longrightarrow \frac{\partial}{\partial Z'^{\mathcal{A}}} = (X^{-1})_{\mathcal{A}}^{\ \mathcal{B}}\frac{\partial}{\partial Z^{\mathcal{B}}}, \tag{2.21}$$

where $X^{\mathcal{A}}_{\ \mathcal{B}} = \frac{\partial Z'^{\mathcal{A}}}{\partial Z^{\mathcal{B}}}$, and similarly for anti-holomorphic differentials and derivatives. Tangent vector fields $V, \bar{V}$ of the manifold can be introduced as objects transforming in the same way as the differentials $dZ, d\bar{Z}$. Using the metric we obtain an invariant inner-product $G_{\underline{\mathcal{A}}\mathcal{A}}\bar{V}^{\underline{\mathcal{A}}}V^{\mathcal{A}}$. Similarly the cotangent vectors can be introduced as objects that transform as derivatives. These (co)tangent vectors are used in section 3 to obtain matter representations needed for matter coupling to supersymmetric σ-models.

The discussion of the Kähler geometry has been done entirely in terms of the metric. For most situations we encounter in this thesis this is sufficient, except in section 2.4 where we discuss holonomy anomalies that can arise in supersymmetric σ-models. Before we can define the holonomy of a Kähler manifold we have to introduce the vielbein formalism for complex manifolds [33, 46, 47]. As before we take $\mathcal{A}, \mathcal{B}, \ldots$ and $\underline{\mathcal{A}}, \underline{\mathcal{B}}, \ldots$ to be curved target space indices, that is indices of the local coordinates of the space. Let $a, b, \ldots$ and $\underline{a}, \underline{b}, \ldots$ be the flat tangent space indices that can be used in the tangent space at a point of the Kähler manifold. The vielbeins $e^a_{\mathcal{A}}$ and $\bar{e}^{\underline{a}}_{\underline{\mathcal{A}}}$ determine the frame in the tangent space over any point of the base space. The vielbeins satisfy

$$G_{\underline{\mathcal{A}}\mathcal{A}} = \bar{e}^{\underline{a}}_{\underline{\mathcal{A}}}\eta_{\underline{a}a}e^a_{\mathcal{A}}, \quad \bar{e}^{\mathcal{A}}_a e^a_{\mathcal{B}} = \delta^{\mathcal{A}}_{\mathcal{B}}, \quad e^a_{\mathcal{A}}\bar{e}^{\mathcal{A}}_b = \delta^a_b \tag{2.22}$$

and the Hermitean conjugate relations [48]. In order that parallel transportation of a vector in the tangent space be compatible with the vielbeins, the metric postulate is imposed. It states that the vielbeins are covariantly constant

$$0 = D_{\mathcal{A}} e^a_{\mathcal{B}} = e^a_{\mathcal{B},\mathcal{A}} - \omega_{\mathcal{A}}{}^a{}_b e^b_{\mathcal{B}} - \Gamma^{\mathcal{C}}_{\mathcal{A}\mathcal{B}} e^a_{\mathcal{C}}, \quad 0 = D_{\underline{\mathcal{A}}} e^a_{\mathcal{B}} = e^a_{\mathcal{B},\underline{\mathcal{A}}} - \omega_{\underline{\mathcal{A}}}{}^a{}_b e^b_{\mathcal{B}}. \tag{2.23}$$

There is no $\Gamma^{\mathcal{C}}_{\underline{\mathcal{A}}\mathcal{B}}$ in the second equation, because it has mixed indices and therefore vanishes for a Kähler manifold. From this the spin-connections $\omega_{\mathcal{A}}{}^a{}_b$ and $\omega_{\underline{\mathcal{A}}}{}^a{}_b$ are obtained

$$\omega_{\underline{\mathcal{A}}}{}^a{}_b = \bar{e}^{\mathcal{B}}_b e^a_{\mathcal{B},\underline{\mathcal{A}}}, \quad \omega_{\mathcal{A}}{}^a{}_b = \bar{e}_{\underline{\mathcal{B}}b} e^{\underline{\mathcal{B}}a}{}_{,\mathcal{A}}, \quad \Gamma^{\mathcal{C}}_{\mathcal{A}\mathcal{B}} = \bar{e}^{\mathcal{C}}_a \left(e^a_{\mathcal{A},\mathcal{B}} - \omega_{\mathcal{B}}{}^a{}_b e^b_{\mathcal{A}} \right), \tag{2.24}$$

using that the connection $\Gamma^{\mathcal{C}}_{\mathcal{A}\mathcal{B}}$ is symmetric in the indices $\mathcal{A}\mathcal{B}$. In terms of the spin-connections we obtain for the curvature (2.3)

$$R_{\mathcal{A}\underline{\mathcal{A}}}{}^a{}_b = \omega_{\underline{\mathcal{A}}}{}^a{}_{b,\mathcal{A}} - \omega_{\mathcal{A}}{}^a{}_{b,\underline{\mathcal{A}}} - [\omega_{\mathcal{A}}, \omega_{\underline{\mathcal{A}}}]^a{}_b = e^a_{\mathcal{D}} \bar{e}^{\mathcal{C}}_b \Gamma^{\mathcal{D}}_{\mathcal{A}\mathcal{C},\underline{\mathcal{B}}}. \tag{2.25}$$

The field strength $R_{\mathcal{A}\mathcal{B}}$ and its conjugate can be shown to vanish. Let the Kähler manifold have complex dimension N. On vectors of the tangent space we can consider the $U(N)$ transformations

$$V^a \longrightarrow V'^{a'} = \Lambda^{a'}{}_a V^a, \qquad \bar{V}^{\underline{a}} \longrightarrow \bar{V}'^{\underline{a}'} = \bar{V}^{\underline{a}} \bar{\Lambda}_{\underline{a}}{}^{\underline{a}'} \tag{2.26}$$

where $\bar{\Lambda}_{\underline{a}}{}^{\underline{a}'} = -\eta_{\underline{a}a} \Lambda^a{}_b \eta^{b\underline{a}'}$. They leave the inner product $\langle \bar{V}, V \rangle = \bar{V}^{\bar{a}} \eta_{\bar{a}a} V^a$ invariant. Thus general coordinate invariance is replaced by $U(N)$ invariance using the vielbein. The $U(N)$ symmetry can be thought of as a gauge invariance with respect to the Kähler manifold, the target manifold, as opposed to spacetime. The holonomy group [37] is the subgroup of $U(N)$ that is obtained by parallel transportation of a vector V^a over closed curves C:

$$V \longrightarrow V = e^{\int_C \omega} V \tag{2.27}$$

where $\omega = \omega_{\mathcal{A}} dZ^{\mathcal{A}} + \omega_{\underline{\mathcal{A}}} dZ^{\underline{\mathcal{A}}}$. By considering an infinitesimal closed curve, it can be shown that the local holonomy group is generated by the curvature tensor $R_{\underline{\mathcal{A}}\mathcal{A}} = [D_{\underline{\mathcal{A}}}, D_{\mathcal{A}}]$ as $D_{\mathcal{A}}$ are the generators of parallel transport.

2.2.1 Examples: Sphere and Hyperbolic Space

In this section we have introduced a number of geometrical tools. It is therefore worthwhile to consider two instructive examples, the two-sphere S^2 and the two dimensional hyperbolic space H^2, before continuing the general discussion. These examples will be used in the first part of this thesis to illustrate various aspects of our general discussion. The two spaces can be treated to a large extent on equal footing, but there are a few important differences that we will stress.

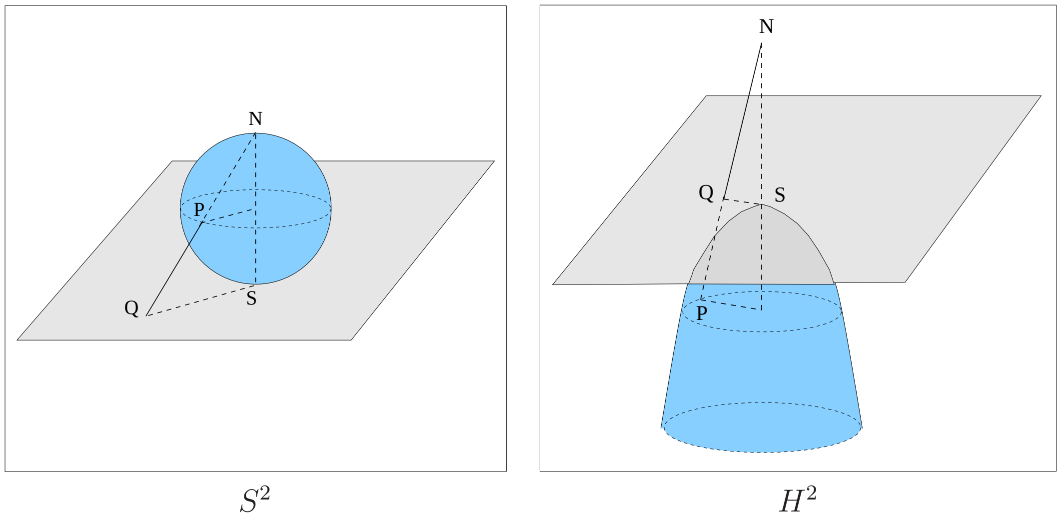


Figure 2.1: Stereo-graphic projection of the sphere S^2 and a connected part of the hyperbolic H^2 space to obtain complex coordinates on these spaces.

Both the two-sphere S^2 and the hyperbolic space H^2 can be embedded in the $(u, v, w) \in \mathbb{R}^3$. These spaces are described by the equations

$$\eta(u^2 + v^2) + w^2 = R^2, \tag{2.28}$$

with $\eta = 1$ for S^2 and $\eta = -1$ for H^2.

The simplest parameterization of the sphere is of course obtained by using spherical coordinates θ, ϕ

$$u = R \sin\theta \cos\phi, \quad v = R \sin\theta \sin\phi, \quad w = R\cos\theta, \tag{2.29}$$

with $0 < \theta < \pi$ and $0 < \phi < 2\pi$. These coordinates do not cover the whole sphere, but by shifting both angles with a non-zero angle θ_0, ϕ_0, another parameterization of the sphere is obtained. The union of these coordinate systems covers the whole sphere. We use a covering of complex coordinate patches, that shows manifestly that the sphere is a Kähler manifold. In fig. 2.1 a stereo-graphic projection on the real plane $\mathbb{R}^2$ is depicted. The projection works as follows: consider the line starting at the north pole N going through point P. Continuing this line until it crosses the tangent plane to the south pole S in Q. The real plane $\mathbb{R}^2$ can also be described as the complex line; then this stereo-graphic projection becomes the mapping

$$(\theta, \phi) \mapsto z_+ = 2R\frac{\sin\theta}{1 - \cos\theta} e^{i\phi}. \tag{2.30}$$

All points of the sphere S^2 are mapped to the complex line with complex coordinate z_+, except the north pole N. To obtain a coordinate patch around N, we rotate the sphere over π round the u-axis: $(\theta, \phi) \mapsto (\pi - \theta, 2\pi - \phi)$. Substituting this in the mapping (2.30) we obtain the stereo-graphic projection with the roles of N and S reversed

$$(\theta, \phi) \mapsto z_- = 2R \frac{\sin\theta}{1 + \cos\theta} e^{-i\phi}. \tag{2.31}$$

These two complex coordinate systems $z_\pm$ cover the sphere completely. To show that the sphere is a complex manifold, we have to show that on the overlap of the two coordinate systems, that is $S^2/\{\text{N,S}\}$ with these coordinates, the transition functions are analytic. To do this we notice that

$$z_+ z_- = (2R)^2. \tag{2.32}$$

From this an analytic mapping $z_- \mapsto z_+(z_-)$ between the two coordinate patches is defined, except at the points N ($z_- = 0$) and S ($z_+ = 0$) on the sphere.
The hyperbolic space is parameterized in close analogy to the sphere by

$$u = R \sinh\theta_\pm \cos\phi_\pm, \quad v = R \sinh\theta_\pm \sin\phi_\pm, \quad w = \pm R \cosh\theta_\pm. \tag{2.33}$$

But as H^2 consists of two disjoint part $H^2_\pm$, we need double coordinates (θ_+, ϕ_+) for H^2_+ and (θ_-, ϕ_-) for H^2_-, with $-\infty < \theta_\pm < \infty$ and $0 < \phi_\pm < 2\pi$. Since H^2_+ is obtained from H^2_- by reflection in the ($w = 0$)-plane we only consider H^2_- from now on and call it H^2. By employing a similar stereo-graphic projection as we used for the two-sphere, see fig. 2.1 again, we obtain the complex coordinate on H^2

$$(\theta, \phi) \mapsto z = 2R \frac{\sinh\theta}{1 + \cosh\theta} e^{i\phi}. \tag{2.34}$$

Since we can cover the whole of H^2 with the coordinates (θ, ϕ) we only need one coordinate patch here. Notice that the complex coordinate z is restricted to the region of the complex-plane for which $|z| < 2R$.

Having discussed the complex coordinates of the sphere and the hyperbolic space, we now introduce metrics on these spaces. The natural metrics on S^2 and H^2 are the induced metrics that are obtained by restricting the metric of $\mathbb{R}^3$ to these spaces. For example in the case of the sphere we have the line-element

$$ds^2 = \left(du^2 + dv^2 + dw^2\right)_{S^2} = R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{2.35}$$

Using the complex coordinates introduced above, the metric for S^2 and H^2 can be written as an invariant line-element

$$ds^2 = G_\sigma dz d\bar{z} = \frac{dz d\bar{z}}{(1 + \eta \bar{z} z/(2R)^2)^2}, \tag{2.36}$$

where as before $\eta = \pm 1$ makes the distinction between S^2 and H^2. The subscript σ is for later use. For the two-sphere we can use both coordinates $z = z_\pm$ in this formula. The metric is form invariant under the coordinate transformation $z_- = (2R)^2/z_+$. Therefore this coordinate transformation is part of the isometry group of the two-sphere.

The manifolds S^2 and H^2 are in fact Kähler since locally the metric can be written as the second mixed derivative (A.14); the Kähler potentials of the different manifolds read

$$K = \eta(2R)^2 \ln\left(1 + \eta \frac{\bar{z}z}{(2R)^2}\right). \tag{2.37}$$

To show that the two-sphere is a Kähler manifold globally, the Kähler potentials have to be related by a sum of an holomorphic plus anti-holomorphic function. In particular the Kähler potentials $K^{\pm}_{S^2}$ for the two coordinate systems on S^2 satisfy

$$K^+_{S^2} = K^-_{S^2} - (2R)^2 \ln\left(\frac{z_-}{2R}\right) - (2R)^2 \ln\left(\frac{\bar{z}_-}{2R}\right). \tag{2.38}$$

Next we discuss globally supersymmetric models in general and come back to this example afterwards to discuss non-linear realizations of the Wess-Zumino model in 2.3.1. In that subsection one can find expressions for the connection and curvature for the two-sphere and the hyperbolic space.

2.3 Globally Supersymmetric Models

In this section we present the machinery to construct $N = 1$ supersymmetric Lagrangeans [49, 50, 51, 52]. For a pedagogical introduction to supersymmetry the reader may consults [13, 14]. For more details on the geometry of supersymmetric models the reader is referred to [35, 47, 53]. All the supersymmetric field theory that is developed here is classical. We introduce a σ-model scale $M_\sigma = f^{-1}$ which is used explicitly only when we need to give a certain object its canonical dimension. We end this section with a discussion of two generalizations of the Wess-Zumino model based on the sphere and the hyperbolic space.

Let $\Sigma^{\mathcal{A}} = (Z^{\mathcal{A}}, \psi_L^{\mathcal{A}}, H^{\mathcal{A}})$ be a set of chiral multiplets, where $Z^{\mathcal{A}}$ is a physical complex scalar, $\psi_L^{\mathcal{A}}$ a chiral fermion and $H^{\mathcal{A}}$ is an auxiliary complex scalar. The index $\mathcal{A}$ enumerates the multiplets in the set. The chiral fermions are assumed to be left-handed throughout this work: $\psi_L^{\mathcal{A}} = \frac{1+\gamma_5}{2}\psi^{\mathcal{A}}$. It is well known that integrating the highest components (the so-called F- and D-terms) of super multiplets over spacetime gives supersymmetric invariants. The kinetic part of the Lagrangean [35] for the chiral multiplets $\Sigma^{\mathcal{A}}$ is given in terms of the D-term of a real composite superfield $\mathcal{K}(\bar{\Sigma}, \Sigma)$ by the following supersymmetric expression

$$\begin{aligned}\mathcal{L}_{\mathcal{K}} \,=\, \mathcal{K}(\bar{\Sigma}, \Sigma)|_D \,=\, -G_{\underline{\mathcal{A}}\mathcal{A}} \left(\partial^\mu \bar{Z}^{\underline{\mathcal{A}}} \partial_\mu Z^{\mathcal{A}} + \bar{\psi}_L^{\underline{\mathcal{A}}} \overleftrightarrow{\not{D}} \psi_L^{\mathcal{A}} - \hat{\bar{H}}^{\underline{\mathcal{A}}} \hat{H}^{\mathcal{A}}\right) \\ + R_{\underline{\mathcal{A}}\mathcal{A}\underline{\mathcal{B}}\mathcal{B}} \, (\bar{\psi}_R^{\underline{\mathcal{A}}} \psi_L^{\mathcal{B}}) \, (\bar{\psi}_L^{\mathcal{A}} \psi_R^{\underline{\mathcal{B}}}).\end{aligned} \tag{2.39}$$

In this expression we used the notation of geometrical objects $G_{\underline{\mathcal{A}}\mathcal{A}}$, $\Gamma^{\mathcal{A}}_{\mathcal{B}\mathcal{C}}$ and $R_{\underline{\mathcal{A}}\mathcal{A}\underline{\mathcal{B}}\mathcal{B}}$ introduced in eqs. (2.1), (2.2) and (2.3). The redefinition of the auxiliary fields $\hat{H}^{\mathcal{A}} = H^{\mathcal{A}} - \Gamma^{\mathcal{A}}_{\mathcal{B}\mathcal{C}}\bar{\psi}^{\mathcal{B}}_{R}\psi^{\mathcal{C}}_{L}$ using the complex connections (2.2) results in a Lagrangean which is written in terms of geometrical objects only. The Kähler covariant derivative is $\not{D}\psi^{\mathcal{A}}_{L} = \gamma^{\mu}D_{\mu}\psi^{\mathcal{A}}_{L} = \gamma^{\mu}\left(\partial_{\mu}\psi^{\mathcal{A}}_{L} + \Gamma^{\mathcal{A}}_{\mathcal{B}\mathcal{C}}\psi^{\mathcal{B}}_{L}\partial_{\mu}Z^{\mathcal{C}}\right)$ and the left-right arrow above the covariant derivative is a short-hand notation for $\bar{\phi}\overleftrightarrow{\not{D}}\psi = \bar{\phi}\gamma^{\mu}D_{\mu}\psi - D_{\mu}\bar{\phi}\gamma^{\mu}\psi$. A few remarks concerning the Lagrangean (2.39) are in order. First of all it is clear from this Lagrangean why $H^{\mathcal{A}}$ are called auxiliary fields: the scalars $H^{\mathcal{A}}$ have no kinetic energy implying that they are not dynamical. Secondly, one of the many consequences of supersymmetry, the kinetic terms for both the physical scalars $Z^{\mathcal{A}}$ and the chiral fermions $\psi^{\mathcal{A}}_{L}$ are determined by the same metric $G_{\underline{\mathcal{A}}\mathcal{A}}$. Since we have expressed the Lagrangean (2.39) in terms of geometrical objects only it is possible to extend the definition of a supersymmetric model to any Kähler manifold even if it is not topologically trivial.

In addition one can write down a Lagrangean determined by the F-term of a holomorphic superfield $W(\Sigma)$ of chiral superfields, called the superpotential,

$$\mathcal{L}_{W} \;=\; [W(\Sigma)]_{F} = \frac{1}{2}W_{,\mathcal{A}}H^{\mathcal{A}} - \frac{1}{2}W_{,\mathcal{A}\mathcal{B}}\bar{\psi}^{\mathcal{A}}_{R}\psi^{\mathcal{B}}_{L} + \text{h.c.} \tag{2.40}$$

The second term of this Lagrangean is used to obtain Yukawa interactions between scalars and chiral fermions.

It follows from the Lagrangean (2.39) that the symmetries of this supersymmetric model are given by the isometries (2.8) of the metric $G_{\underline{\mathcal{A}}\mathcal{A}}$ which leave the superpotential (2.40) invariant. This follows as all geometrical objects in (2.39) are derived from the metric and are therefore invariant as well. In fact constraining the superpotential to be invariant is too strong: W may transform with a phase factor which does not depend on the fields, because this can be compensated by a chiral rotation of the fermions to leave the Yukawa couplings invariant. This rotation of chiral fermions is known as R-symmetry. R-symmetry is broken if the superpotential does not transform homogeneously.

The infinitesimal transformation rules $\delta_{i}\Sigma^{\mathcal{A}} = \mathcal{R}^{\mathcal{A}}_{i}(\Sigma)$ of the chiral multiplet $Z^{\mathcal{A}}$ are completely determined by the Killing vectors $\mathcal{R}^{\mathcal{A}}_{i}(Z)$ if we require them to respect supersymmetry. In components the transformation rules read

$$\begin{aligned}
\delta_{i}Z^{\mathcal{A}} &= \mathcal{R}^{\mathcal{A}}_{i}(Z),\\
\delta_{i}\psi^{\mathcal{A}}_{L} &= \mathcal{R}^{\mathcal{A}}_{i,\mathcal{B}}(Z)\psi^{\mathcal{B}}_{L},\\
\delta_{i}H^{\mathcal{A}} &= \mathcal{R}^{\mathcal{A}}_{i,\mathcal{B}}(Z)H^{\mathcal{B}} - \mathcal{R}^{\mathcal{A}}_{i,\mathcal{B}\mathcal{C}}(Z)\bar{\psi}^{\mathcal{B}}_{R}\psi^{\mathcal{C}}_{L}.
\end{aligned} \tag{2.41}$$

If some of the internal symmetries are local, the partial derivatives ∂_{μ} in eq. (2.39) and in the Kähler covariant derivative D_{μ} have to be replaced by gauge covariant ones $\mathcal{D}_\mu$ given by

$$\begin{aligned} \partial_\mu Z^{\mathcal{A}} \quad &\longrightarrow \quad \mathcal{D}_\mu Z^{\mathcal{A}} = \partial_\mu Z^{\mathcal{A}} - A^i_\mu \mathcal{R}^{\mathcal{A}}_i , \\ D_\mu \psi^{\mathcal{A}}_L \quad &\longrightarrow \quad \mathcal{D}_\mu \psi^{\mathcal{A}}_L = \partial_\mu \psi^{\mathcal{A}}_L - A^i_\mu \mathcal{R}^{\mathcal{A}}_{i,\mathcal{B}} \psi^{\mathcal{B}}_L + D_\mu Z^{\mathcal{C}} \Gamma^{\mathcal{A}}_{\mathcal{CB}} \psi^{\mathcal{B}} , \end{aligned} \tag{2.42}$$

where A^i_μ are the gauge fields corresponding to the local symmetries. They are components of the vector multiplets $V^i = (A^i_\mu, \lambda^i, D^i)$, with λ^i representing the gauginos and D^i the real auxiliary fields.

After the introduction of the gauge fields in the Lagrangean (2.39) via the covariant derivatives (2.42), the σ-model Lagrangean itself is not invariant under supersymmetry transformations anymore. This can be resolved by adding the terms [49]

$$\Delta \mathcal{L}_{\mathcal{K}} \;=\; 2\, G_{\underline{\mathcal{A}}\mathcal{A}} \left(\bar{\mathcal{R}}^{\underline{\mathcal{A}}}_i \bar{\lambda}^i_R \psi^{\mathcal{A}}_L + \mathcal{R}^{\mathcal{A}}_i \bar{\psi}^{\underline{\mathcal{A}}}_L \lambda^i_R \right) - D^i \left(\mathcal{M}_i + \xi_i \right) \tag{2.43}$$

to the Lagrangean (2.39). Here we have included a possible Fayet-Iliopoulos term for any $U(1)$ factor in the group G [54]. This is allowed as D^i are the highest components of vector multiplets and they are inert when they correspond to a $U(1)$ factor symmetry.

The kinetic terms for these vector multiplets take the form [55, 56] of an F-term of the composite chiral superfield

$$\begin{aligned} \mathcal{L}_f \;=\; \left[f_{ij}(\Sigma) W^i(V)\, W^j(V) \right]_F &= \frac{1}{2} f_{ij} \left(-\bar{\lambda}^i_R \overleftrightarrow{\not{D}} \lambda^i_R - \frac{1}{2} F^{i-} \cdot F^{j-} + \frac{1}{2} D^i D^j \right) \\ &+ \frac{1}{2} f_{ij,\mathcal{A}} \left(-\sigma \cdot F^{i-} + i D^i \right) \bar{\psi}^{\mathcal{A}}_R \lambda^j_L - \frac{1}{4} f_{ij,\mathcal{A}} H^{\mathcal{A}} \lambda^i_R \lambda^j_L \\ &+ \frac{1}{4} f_{ij,\mathcal{AB}} (\bar{\psi}^{\mathcal{A}}_R \psi^{\mathcal{B}}_L)\, (\bar{\lambda}^i_R \lambda^j_L) + \text{h.c.}, \end{aligned} \tag{2.44}$$

where the $f_{ij}(\Sigma)$ is a holomorphic function that transforms covariantly under the isometries, called the gauge-kinetic function. The multiplets W^i are a particular kind of chiral spinor multiplets, that have as lowest component the chiral spinors λ^i_L and are obtained from the vector multiplets by

$$W^i_L(V) \;=\; \left(\lambda^i_L, \frac{1}{2} \left(-\sigma \cdot F^{i-} + i D^i \right), \not{D} \lambda^i_R \right) . \tag{2.45}$$

Here the anti-self-dual field strength is defined as $F^{i-}_{\mu\nu} = \frac{1}{2} \left(F^i_{\mu\nu} - \tilde{F}^i_{\mu\nu} \right)$ with $\tilde{F}^i_{\mu\nu}$ the dual tensor of $F^i_{\mu\nu}$. The covariant derivative acting on the gauginos is defined in the adjoint representation. The standard form of the function f_{ij} is $f_{ij}(Z) = \sigma(Z) \eta_{ij}$ where η_{ij} is the Killing metric defined in eq. (2.14) and $\sigma(Z)$ is an invariant holomorphic scalar coefficient. The indices $i, j, \ldots$ run over the gauged part of the isometries only. The real part of f_{ij} can be interpreted as the gauge coupling constant $1/g^2$. When a direct product group of subgroups is gauged, there are as many different coupling constants as there are subgroups.

2.3.1 Examples: Non-Linear Wess-Zumino Models

We now describe how on the two Kähler manifolds discussed in subsection 2.2.1 (the two-sphere S^2 and the two dimensional hyperbolic space H^2) we can construct a supersymmetric field theory. These models can be thought of as non-linear extensions of the Wess-Zumino model [57] for one chiral multiplet. The Kähler potentials (2.37) for the sphere and the hyperbolic space can be written as

$$K_\sigma(\bar{z}, z) = \frac{1}{\eta f^2} \ln(1 + \eta f^2 \bar{z} z). \tag{2.46}$$

where z is now interpreted as a σ-model field and the σ-model scale $f^{-1} = M_\sigma = 2R$ is related to the radius of curvature. The notation K_σ is for later use. The metric (2.36), the connection and the curvature are given by

$$G_\sigma = \chi^2, \qquad \Gamma = -2\eta f^2 \chi \bar{z}, \qquad R = -2\eta f^2 \chi^4. \tag{2.47}$$

using the submatrix $\chi = (1 + \eta f^2 \bar{z} z)^{-1}$. The resulting Lagrangean is

$$\mathcal{L}_{K_\sigma} \;=\; -G_\sigma \left(\partial^\mu \bar{z} \partial_\mu z + \bar{\psi}_L \overleftrightarrow{\not{D}} \psi_L - \hat{\bar{H}} \hat{H} + 2\eta f^2 G_\sigma (\bar{\psi}_R \psi_L)\, (\bar{\psi}_L \psi_R) \right) \tag{2.48}$$

If we take $f \to 0$, then the Kähler potential becomes $K = \bar{z} z$ and the Lagrangean reduces to the renormalizable Wess-Zumino model [57].

Next we turn to the symmetries of this model. The infinitesimal isometries of the metric (2.36)

$$\delta(\theta) z = \frac{1}{f}\epsilon + \eta f\, z \bar{\epsilon} z + 2i\theta_Y z, \qquad \delta(\theta)\bar{z} = \frac{1}{f}\bar{\epsilon} + \eta f\, \bar{z}\epsilon\bar{z} - 2i\theta_Y \bar{z} \tag{2.49}$$

form a non-linear representation of the algebra of $SU_\eta(1,1)$. When $\eta = 1$ then this is the algebra of the compact group $SU(2)$ and when $\eta = -1$ of the non-compact group $SU(1,1)$. Generators of the different transformations are Y of the $U_Y(1)$ subgroup, with the related parameter θ_Y, and X and $\bar{X}$ of the remaining $SU_\eta(1,1)$ symmetries, with related parameters $\theta_X = \bar{\epsilon}$ and $\theta_{\bar{X}} = \epsilon$. For all parameters we write collectively $\theta = (\epsilon, \bar{\epsilon}, \theta_Y)$. The above transformations form a non-linear representation of the Lie algebra:

$$[Y, X] = 2X, \quad [Y, \bar{X}] = -2\bar{X}, \quad [\bar{X}, X] = -\eta Y \tag{2.50}$$

where we only give the non-vanishing commutators. In an explicit matrix representation the generators take the form

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{X} = \begin{pmatrix} 0 & 0 \\ \eta & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.51}$$

Here Y is chosen Hermitean and $\bar{X} = \eta X^\dagger$ is the conjugate of X. By computing $[\delta(\theta^1), \delta(\theta^2)] = \delta(\theta^3)$ using both the transformation rules (2.49) and the abstract algebra (2.50) with $\delta(\theta) = i\epsilon\bar{X} + i\bar{\epsilon}X + i\theta_Y Y$, we find that

$$\begin{aligned} \epsilon^3 &= 2i\Big(\epsilon^1\theta_Y^2 - \epsilon^2\theta_Y^1\Big), \\ \bar{\epsilon}^3 &= -2i\Big(\bar{\epsilon}^1\theta_Y^2 - \bar{\epsilon}^2\theta_Y^1\Big), \end{aligned} \qquad \theta_Y^3 = -\eta i\Big(\epsilon^1\bar{\epsilon}^2 - \epsilon^2\bar{\epsilon}^1\Big). \tag{2.52}$$

This shows that the isometries (2.49) are indeed a representation of the algebra (2.50). Under these isometries K_σ transforms as

$$\delta(\theta)K_\sigma(z,\bar{z}) = F(z;\theta) + \bar{F}(\bar{z};\theta), \qquad F(z;\theta) = \frac{1}{\eta f^2}\left(\eta f z\bar{\epsilon} + i\theta_Y\right). \tag{2.53}$$

according to eq. (2.15). The first term in the brackets in $F(z;\theta)$ follows from a direct computation of δK_σ using the isometries (2.49). The second term is determined by demanding that $F(z;\theta)$ transforms according to eq. (2.16) in the adjoint of the algebra (2.50)

$$\delta(\theta^1)F(z;\theta^2) - \delta(\theta^2)F(z;\theta^1) = F(z;\theta^3), \tag{2.54}$$

where the parameters θ^3 are given by (2.52).

2.4 Anomalies of Supersymmetric σ-Models

In most of this thesis we consider classical field theories only. However we want to make sure that the quantum theory does not destroy the structure of the classical theory determined by symmetries. Anomalies are violations of classical symmetries of a theory by quantum effects. Textbook introductions to the subject of anomalies can be found in refs. [2, 3, 8]. Refs. [28, 58, 59] are more technical reviews of gauge and gravitational anomalies. Various aspects of σ-model anomalies are discussed in refs. [33, 60, 61].

For supersymmetric non-linear σ-models coupled to $N = 1$ supergravity there are many symmetries that can develop an anomaly. (Although this chapter is primarily devoted to globally supersymmetric models, gravitational interactions are considered here as well. In section 3.5 we discuss supergravity in more detail.) Anomalies in chiral theories in $D = 4$ dimensions arise from triangle diagrams as depicted in figure 2.2, where the chiral fermions run around the loop. The vector fields that couple at the corners of the triangle can be physical gauge bosons, when a local symmetry is involved, or composite fields. One can distinguish between pure and mixed anomalies: pure anomalies are anomalies where just one symmetry hence one type of vector fields is concerned. Mixed anomalies involve more than one symmetry. Apart from these perturbative anomalies, we briefly discuss a related topological obstruction to obtain an effective action that can

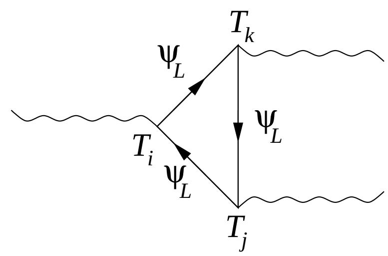


Figure 2.2: In a theory with chiral fermions the triangle diagram where the chiral fermions run in the loop are the prime source of anomalies in $D = 4$ dimensions.

be interpreted as a function when the fermions have been integrated out. This obstruction is called the global σ-model anomaly.

Let us list the symmetries in (local) supersymmetric non-linear σ-models that can develop anomalies. First of all we have the internal symmetries of the supersymmetric model; from a geometrical point of view they are the isometries of the Kähler manifold underlying the σ-model. The isometries can be global or local symmetries with respect to spacetime. The next symmetry of the classical theory is the holonomy $\mathcal{H}$ when the Lagrangean is written in terms of the vielbeins instead of the metrics. Anomalies in the holonomy can be thought of as gravitational anomalies with respect to the geometry of the Kähler manifold. In addition we can have spacetime gravitational anomalies as well. A final invariance of the classical theory is the so-called Kähler invariance, which for global supersymmetric case means that the Kähler potential is determined up to the sum of a holomorphic function and its conjugate. For local supersymmetry the superpotential is required to transform under this symmetry as well. Anomalies in this classical invariance are called Kähler anomalies.

Anomalous internal gauge symmetries or spacetime gravitational anomalies are disastrous for the theory. The reason is that non-physical degrees of freedom, the time-like polarizations of gauge fields, enter the S-matrix and develop poles. This leads to a breakdown of unitarity and makes the quantum theory meaningless. The only way to cancel anomalous internal gauge symmetries is by a Wess-Zumino counter term [27]. If we cannot or do not want to resort to a Wess-Zumino counter term, we have to make sure that the chiral fermion content of the model is such that all contributions to anomalous triangle diagrams cancel among themselves for all local spacetime and internal gauge symmetries. Anomalies in the global internal symmetries or isometries do not lead to breakdown of unitarity; hence it is not a fundamental problem if such anomalies exist.

We now discuss holonomy, gauge and spacetime gravitational anomalies in more detail. To study anomalies carefully, we consider the effective action W_F

that is obtained when the chiral fermions are integrated out

$$e^{iW_F} = \int \mathcal{D}\psi_L^a \mathcal{D}\bar{\psi}_L^{\underline{a}} \exp\left\{-i\int d^4x \eta_{\underline{a}a} \bar{\psi}_L^{\underline{a}} \overleftrightarrow{\not{D}} \psi_L^a\right\}. \tag{2.55}$$

Several remarks are in order here: we have introduced fermions with flat target-space indices by using the vielbeins (2.22) ($\psi_L^a = e_{\mathcal{A}}^a \psi_L^{\mathcal{A}}$, etc.). As the metric $\eta_{\underline{a}a}$ of these fermions is equal to the identity, we do not have to include a compensating determinant in the measure of the path integral. The covariant derivative on the fermion ψ_L^a is given by

$$D_\mu \psi_L^a = \partial_\mu \psi_L^a - C_\mu{}^a{}_b \psi_L^a \tag{2.56}$$

with the connection C_μ consisting of three parts

$$C_\mu{}^a{}_b = \frac{1}{4}\omega_\mu^{mn}\Sigma_{mn}\delta^a{}_b + A_\mu^i \mathcal{R}_i{}^a{}_b + B_\mu{}^a{}_b + \frac{1}{4}R_{\underline{\mathcal{A}}\mathcal{A}\underline{\mathcal{B}}\mathcal{B}}(\bar{\psi}_L^{\underline{\mathcal{A}}}\gamma_\mu\psi_L^{\mathcal{A}})e^{a\mathcal{B}}\bar{e}_b^{\underline{\mathcal{B}}}. \tag{2.57}$$

The first term contains the spacetime spin-connection ω_μ^{mn}, the second term the gauge internal symmetries (where $\mathcal{R}_i{}^a{}_b$ is obtained from $\mathcal{R}_{i,\mathcal{B}}^{\mathcal{A}}$ contracted with the (inverse) vielbeins) with gauge potential A_μ^i and B_μ is the composite vector potential

$$B_\mu{}^a{}_b = (\partial_\mu Z^{\mathcal{A}})\omega_{\mathcal{A}}{}^a{}_b + (\partial_\mu \bar{Z}^{\underline{\mathcal{A}}})\omega_{\underline{\mathcal{A}}}{}^a{}_b \tag{2.58}$$

expressed in terms of the spin-connections (2.24) of the Kähler manifold. Finally using a Fierz-reordering $R_{\underline{\mathcal{A}}\mathcal{A}\underline{\mathcal{B}}\mathcal{B}}(\bar{\psi}_L^{\underline{\mathcal{A}}}\psi_R^{\mathcal{B}})(\bar{\psi}_R^{\underline{\mathcal{A}}}\psi_L^{\mathcal{B}}) = \frac{1}{2}R_{\underline{\mathcal{A}}\mathcal{A}\underline{\mathcal{B}}\mathcal{B}}(\bar{\psi}_L^{\underline{\mathcal{A}}}\gamma^\mu\psi_L^{\mathcal{A}})(\bar{\psi}_L^{\underline{\mathcal{B}}}\gamma_\mu\psi_L^{\mathcal{B}})$, the four-fermion terms of eq. (2.39) are absorbed in the definition of the connection C_μ. This does not affect the anomaly analysis as it does not transform inhomogeneously.

The anomalous behavior [62, 63] is given by the non-invariance of the fermionic effective action W_F

$$\delta_\Lambda W_F = -\frac{i}{24\pi^2}\int d^4x\, \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left[\Lambda\partial_\mu\Big(C_\nu\partial_\rho C_\sigma - \frac{1}{2}C_\nu C_\rho C_\sigma\Big)\right], \tag{2.59}$$

where Λ depends on the symmetry under consideration: for local spacetime Lorentz-transformations, gauged internal symmetries and holonomy we have as parameters $\Lambda = \frac{1}{4}\Lambda^{mn}\Sigma_{mn}$, $\Lambda^a{}_b = \Lambda^i \mathcal{R}_i{}^a{}_b$ and $\Lambda \in \mathcal{H} \subset U(N)$, respectively. The parameter Λ of the holonomy restricted to the global symmetries [48] is given by

$$\Lambda^a{}_b = \Lambda^i\left[D_b\mathcal{R}_i{}^a + \Big(R_i^{\mathcal{A}}\omega_{\mathcal{A}}\mathcal{R}_i^{\underline{\mathcal{A}}}\omega_{\underline{\mathcal{A}}}\Big)^a{}_b\right]. \tag{2.60}$$

It can be shown [62, 64] that the integrated anomaly $\delta_\Lambda W_F$ is proportional to the symmetric trace

$$d_{ijk} = \mathrm{Tr}\left(\{T_i, T_j\}T_k\right). \tag{2.61}$$

Here the T_i refer to the generators of all continuous classical symmetries.

Even if we do not include any gauge fields or spacetime spin-connections in the effective action (2.55), it is not clear that we can interpret (2.55) as function of the scalar boson configuration. As was shown in refs. [61, 65, 66, 67] this is only the case when the integer

$$\nu = \int_{S^2 \times S^4} \hat{\phi}^* ch_3 T\mathfrak{M}_\sigma = \int_{S^2 \times S^4} \hat{\phi}^* \mathrm{Tr} R^3 \tag{2.62}$$

is equal to zero. Here $\hat{\phi}^*$ is the pull back of the third Chern character of $T\mathfrak{M}_\sigma$ on $S^2 \times S^4$. (See ref. [37, 39] for the definition of the Chern character.) When the topological invariant ν is non-zero, e^{iW_F} does not make sense as an effective action: the value of it depends on coordinate choices. But e^{iW_F} can still be interpreted as a section of a line bundle. (Line bundles are discussed in appendix B.)

In the remainder of this thesis we take the following approach to anomalies [68, 69, 70]. We start with a non-linear supersymmetric σ-model based on a homogeneous Kählerian manifold that in general has isometry anomalies and mixed isometry-spacetime gravity anomalies. Next we determine a representation of chiral fermions such that the anomalies cancel. These chiral fermions contain at least the super partners of the coordinates of the manifold. To preserve supersymmetry all the other chiral fermions needed for anomaly cancellation have to be included in chiral multiplets. These additional multiplets, called matter multiplets, have to satisfy various properties which are discussed in the next chapter 3. We do not resort to Wess-Zumino counter terms to cancel anomalies since they do not (always) exist for the models based on Kählerian cosets we consider later. In section 4.6 we discuss anomaly issues for coset spaces in more detail.

Chapter 3

Supersymmetric Matter Coupling

3.1 Introduction

The present chapter discusses different aspects of supersymmetric matter coupling to supersymmetric non-linear σ-models. As explained in section 2.4 we will try to cancel anomalies due to chiral superpartners of the coordinates of a Kähler manifold by adding additional chiral matter multiplets. Therefore it is necessary to investigate which options we have to construct matter multiplets that respect all isometries of the original model.

In section 3.2 we define the notions σ-model and matter multiplets more precisely. The next section 3.3 gives a number of different types of matter coupling that can be interpreted as tangent vectors, tensors and non-trivial singlets from a geometrical perspective of the original Kähler manifold. The singlet representation plays an essential role in the remainder of this thesis, since it makes more flexible charge assignments possible. The matter representations can be interpreted as sections of bundles over the original Kähler manifold; in particular the non-trivial singlet is a section of a complex line bundle. Appendix B gives some mathematical background on bundles. For topologically non-trivial manifolds a consistency condition of the complex line bundle restricts the charge assignments. To illustrate the procedure of matter coupling to non-linear σ-models we return in section 3.4 to our examples of the Wess-Zumino model on the sphere and the hyperbolic space.

Supersymmetric matter coupling to non-linear σ-models in the context of supergravity has some consequences which are discussed in section 3.5. We first review the construction of locally supersymmetric Lagrangeans for chiral and vector multiplets. We explain why the σ-model multiplets cannot have a Weyl weight. We also discuss the changes to the supergravity Lagrangean when non-linear symmetries are gauged. Coupling a non-linear σ-model based on a topologically non-trivial Kähler manifold to supergravity leads to additional requirements on this manifold and Newton's constant may have to be quantized. We end our discussion of supergravity with an analysis of the scalar potential.

In the kinetic Lagrangean of supersymmetric non-linear σ-models there is often mixing between different irreducible representations. In addition the fermions often do not transform homogeneously under isometry transformations. In the final section 3.6 we explain how these problems can be solved by using non-holomorphic transformations.

3.2 Non-Linear Isometry Representations

Until this point we have treated all chiral multiplets $\Sigma^{\mathcal{A}} = \left(\Phi^\alpha,\, \Psi^A\right)$ on equal footing; we now divide the chiral multiplets into two sets $\{\Phi^\alpha\}$ and $\{\Psi^A\}$. The elements of these sets are classified by transformation properties under the isometries. The chiral multiplets $\Phi^\alpha = (z^\alpha, \psi_L^\alpha, h^\alpha)$ transforming non-linearly into themselves under a part of the isometries are called *σ-model* multiplets. (In string theory applications these multiplets are often referred to as moduli superfields [71, 72].) The *matter* multiplets $\Psi^A = \left(x^A, \chi_L^A, f^A\right)$ are chiral multiplets that transform linearly into themselves under all isometries, but possibly with σ-model-field dependent parameters. The transformations (2.41) of σ-model and matter multiplets have the form [68, 69, 73, 74]

$$\delta_i \Phi^\alpha = R_i^\alpha(\Phi), \qquad \delta_i \Psi^A = R^A_{i\,B}(\Phi)\Psi^B \tag{3.1}$$

according to the definitions above. The Killing vectors (3.1) for the σ-model and matter multiplets satisfy

$$R^\beta_{[i} R^\alpha_{j],\beta} = f_{ij}{}^k R^\alpha_k, \qquad R^B_{[i\,C} R^A_{j]\,B} + R^\beta_{[i} R^A_{j]\,C,\beta} = f_{ij}{}^k R^A_{k\,C}. \tag{3.2}$$

The components of the σ-model multiplets Φ^α transform according to (2.41) but with $Z^{\mathcal{A}}$ replaced by z^α. The matter multiplets also transform according to (2.41), but because $\delta_i\Psi^A$ depends on both Ψ^A and Φ^α we get two contributions in the second equation of (3.2). The transformation rules for components of the matter multiplets $\delta_i\Psi^A = \delta_i\left(x^A, \chi_L^A, f^A\right)$ are obtained by expanding the chiral superfields in their components

$$\begin{aligned}
\delta_i x^A &= R^A_{i\,B} x^B, \\
\delta_i \chi_L^A &= R^A_{i\,B}\chi_L^B + R^A_{i\,B,\beta} x^B \psi_L^\beta, \\
\delta_i f^A &= R^A_{i\,B} f^B + R^A_{i\,B,\beta}\left(x^B h^\beta - 2\bar\chi_R^B \psi_L^\beta\right) - R^A_{i\,B,\beta\gamma} x^B \bar\psi_R^\beta \psi_L^\gamma.
\end{aligned} \tag{3.3}$$

Notice that the chiral matter fermions χ_L^A do not transform homogeneously into themselves if the transformations $R^A_{i\,B}$ depend on the σ-model fields. As the fermions are physical particles, one would like to be able to redefine the fermions in such a way that

$$\delta_i \hat{\chi}_L^A = R_{i\,B}^A \hat{\chi}_L^B. \tag{3.4}$$

We describe in section 3.6 how this can be done for an arbitrary σ-model/matter system described by a Kähler potential $K(\bar{z}, \bar{x}; z, x)$, see eq. (3.58). For the matter auxiliary fields it is not a problem that they do not transform linearly into themselves, as these auxiliary fields can be eliminated by their equations of motion from the theory anyway.

The definition of matter coupling given above applies only locally, hence it is insufficient for topologically non-trivial Kähler manifolds $\mathfrak{M}_\sigma$. The scalar components of the σ-model and matter multiplets have to parameterize a Kähler manifold again, because only with a Kähler manifold we can associate a supersymmetric model again. This is guaranteed if the matter multiplets are sections of bundles over the original Kähler manifold $\mathfrak{M}_\sigma$. The mathematical details of bundles can be found in appendix B, here we just sketch the idea. If a manifold is topologically non-trivial one needs more than one coordinate patch to cover the whole space. Within one coordinate patch $U_{(a)}$ a section $x^{(a)}$ is simply a function of that part of the manifold. As the manifold was formed by gluing of the various coordinate patches, we have to glue the local functions on the coordinate patches as well. It is not necessary that these functions $x^{(a)}, x^{(b)}$ on different coordinate patches $U_{(a)}, U_{(b)}$ are identical on the overlaps, but they have to be related by a transition function $g^{(ab)} \in \mathrm{G}$ as $x^{(a)} = g^{(ab)} x^{(b)}$, where the group G is called the structure group of the bundle. This can be done consistently over the whole manifold when the transition functions $g^{(ab)}$ satisfy the cocycle conditions, given in eq. (B.1) of appendix B.

We finish this section by fixing the notation for the general considerations below. In the following we denote the Kähler potential for all physical σ-model multiplets Φ^α by $K_\sigma(\bar{\Phi}, \Phi)$, and the Kähler potential for all physical matter multiplets Ψ^A by $K_m(\bar{\Phi}, \bar{\Psi}; \Phi, \Psi)$. For all chiral multiplets we write $\Sigma^{\mathcal{A}} = (\Phi^\alpha, \Psi^A)$. The Kähler potential for the combined σ-model and matter system is given by

$$K(\bar{\Phi}, \bar{\Psi}; \Phi, \Psi) = K_\sigma(\bar{\Phi}, \Phi) + K_m(\bar{\Phi}, \bar{\Psi}; \Phi, \Psi) \tag{3.5}$$

the sum of these two Kähler potentials. In the discussion of the superpotential (2.40), it is convenient to introduce a *compensating* superpotential $w(\Sigma)$: a dimensionless composite chiral superfield which transforms as $\delta_i w = q f^2 F_i w$ under the internal symmetries, with q a real number. We will later see that q is often quantized. A *covariant* superpotential $\mathcal{W}$ is defined by

$$\mathcal{W}(\Sigma) = w(\Sigma) W(\Sigma), \tag{3.6}$$

combining the *invariant* superpotential W, as in eq. (2.40), with the compensating superpotential w introduced above. It transforms under the isometries as

$$\delta_i \mathcal{W} = q f^2 F_i \mathcal{W}. \tag{3.7}$$

With such a holomorphic function $\mathcal{W}$, an *invariant* Kähler potential can be defined in terms of the physical fields only

$$\mathcal{K}(\bar{\Sigma}, \Sigma) = K(\bar{\Sigma}, \Sigma) - \frac{1}{qf^2} \ln |f^2\, \mathcal{W}(\Sigma)|^2. \tag{3.8}$$

3.3 Matter Representations

The previous section discussed matter coupling in general. We now construct vector, tensor and non-trivial singlet matter representations coupled to a non-linear σ-model based on the Kähler manifold $\mathfrak{M}_\sigma$. In this section the focus is mostly on the effects of infinitesimal isometry transformations and the definition of invariant Kähler potentials for the matter representations, but we comment on bundle aspects as well. For a non-trivial singlet representation we find the bundle consistency leads to charge quantization. We end this section by giving an expression for the Killing potential for general matter coupling.

Following [73], the matter coupling to a Kähler manifold is discussed in a purely geometrical fashion and hence can be applied to any supersymmetric σ-model. This construction is called covariant matter coupling. The starting point is a σ-model Kähler potential $K_\sigma(\bar{\Phi}, \Phi)$ of the σ-model multiplets $\Phi^\alpha, \bar{\Phi}^{\underline{\alpha}}$, the scalar components of which parameterize a Kähler manifold $\mathfrak{M}_\sigma$. Under the isometries, these multiplets transform according to the first equation in (3.1) and the Kähler potential is covariant

$$\delta_i K_\sigma(\bar{\Phi}, \Phi) = F_i(\Phi) + \bar{F}_i(\bar{\Phi}). \tag{3.9}$$

As the metric G_σ defines an invariant line element $ds^2 = d\bar{z}^{\underline{\alpha}} G_{\sigma\, \underline{\alpha}\alpha} dz^\alpha$, it follows that a matter multiplet $\Psi^\alpha = (x^\alpha, \chi_L^\alpha, f^\alpha)$ which transforms as a tangent vector

$$\delta_i \Psi^\alpha = R^\alpha_{i,\beta}(\Phi) \Psi^\beta, \tag{3.10}$$

has an invariant Kähler potential [73, 74, 69] given by

$$K_1(\bar{\Psi}, \Psi; \bar{\Phi}, \Phi) = \bar{\Psi}^{\underline{\alpha}} G_{\sigma\, \underline{\alpha}\alpha} \Psi^\alpha. \tag{3.11}$$

By taking the D-term of this real superfield a kinetic Lagrangean for the multiplet Ψ^α is obtained. With the subscript 1 we indicate that this is the coupling of a rank-one tensor (a vector) to the σ-model.

By taking tensor products of p such vectors one can built a rank-p-tensor chiral multiplet coupled to the σ-model. Introducing multi-indices $A = (\alpha_1 \ldots \alpha_p)$, we write the components of a rank-p tensor as $\Psi^{\alpha_1 \ldots \alpha_p} \equiv \Psi^A = (x^A, \chi_L^A, f^A)$. Under the action of the isometries this tensor chiral multiplet transforms as

$$\delta_i \Psi^A = R_{i\,B}^A \Psi^B \equiv \sum_{k=1}^{p} R^{\alpha_k}_{i,\,\gamma}(\Phi)\, \Psi^{\alpha_1 \ldots \gamma \ldots \alpha_p}. \tag{3.12}$$

It is straightforward to check, that these transformations define a representation of the Lie algebra (3.2). It is possible to construct irreducible representations of the linear isometries by (anti)-symmetrizations and by taking traces. Invariant Lagrangeans for the superfield Ψ^A are easily constructed by adding diffeomorphism invariant terms to the Kähler potential, the simplest one being

$$K(\bar{\Phi}, \bar{\Psi}; \Phi\Pi) = G_{\underline{A}A}\, \bar{\Psi}^{\underline{A}}\Psi^A \equiv G_{\sigma\, \underline{\alpha}_1\alpha_1} \ldots G_{\sigma\, \underline{\alpha}_p\alpha_p}\, \bar{\Psi}^{\underline{\alpha}_1\cdots\underline{\alpha}_p}\Psi^{\alpha_1\cdots\alpha_p}. \tag{3.13}$$

Other possibilities involve e.g. contractions with curvature components, and terms of higher order in $\bar{\Psi}$ and Ψ [73]. It is not difficult to define the vector matter representation globally, because we can interpret Ψ^α as a section of the tangent bundle. Similarly the tensor Ψ^A is a section of the tensor product of p copies of the tangent bundle. In addition we can also use the cotangent bundle to obtain matter couplings.

It is also possible to couple a singlet chiral superfield $\Omega = (s, \chi_L, \mathrm{f})$ non-trivially to a σ-model [69]. The singlet chiral multiplet Ω transforms as

$$\delta_i\Omega = -f^2 F_i(\Phi)\Omega, \tag{3.14}$$

which forms a representation when the holomorphic functions F_i are such that

$$\delta_{[i}F_{j]} = f_{ij}{}^k F_k. \tag{3.15}$$

We argued above eq. (2.16) that it is always possible to obtain such functions F_i. The transformations of Ω and the Kähler potential (3.9) show that the real composite superfield

$$\bar{\Omega}^q\Omega^q\, e^{qf^2\, K_\sigma(\bar{\Phi},\Phi)} \tag{3.16}$$

with $q \in \mathbb{Z}$ is invariant under the internal symmetries generated by the Lie-algebra of the holomorphic Killing vectors. This construction can also be employed to describe Berezin quantization [1] on homogeneous Kähler manifolds [75, 76].

The non-trivial singlet representation has to be interpreted as a section of a complex line bundle over the manifold $\mathfrak{M}_\sigma$. As is shown in appendix B the consistency of a line bundle requires that the Kähler form (2.5) of the Kähler potential $qf^2\, K_\sigma$ in the exponential in eq. (3.16) has to satisfy

$$q\, f^2 \int_{C_2} \omega(K_\sigma) \in 2\pi\mathbb{Z} \tag{3.17}$$

when it is integrated over a two-cycle (for example a smooth closed two-dimensional surface, like the sphere). We see that the rescaling charge q is quantized, in particular there is a minimal positive rescaling charge. Manifolds that satisfy this condition are called Kähler-Hodge manifolds [38].

[1] Berezin quantization is the construction of the Hilbert space of holomorphic functions on a Kähler manifold.

This singlet representation is very important to obtain more flexibility in the charge assignment of matter representations. The matter representations transforming as the tensor products of the tangent vectors have only positive integer multiples of the charge of the coordinates of the manifold. We take the singlet superfield Ω to be non-propagating and dimensionless. One can use this *compensating* multiplet Ω to rescale any matter multiplet Ψ^A so as to assign arbitrary *rescaling charges* $q^{(A)}$ to it [69] by defining $\Psi'^A = \Omega^{-q^{(A)}}\Psi^A$. With the notation $q^{(A)}$ we indicate that within an irreducible representation the rescaling charges are equal. The transformation rule Ψ'^A becomes

$$\delta_i \Psi'^A = R^A_{i\,B}(\Phi)\Psi'^B + q^{(A)} f^2 F_i(\Phi)\Psi'^A \tag{3.18}$$

and its Kähler potential has to be modified to

$$K_m(\bar{\Psi}', \Psi'; \bar{\Phi}, \Phi) = \bar{\Psi}'^{\underline{A}} G'_{\underline{A}A}\Psi'^A = \bar{\Psi}'^{\underline{A}} e^{-q^{(A)} f^2 K_\sigma} G_{\underline{A}A}\Psi'^A. \tag{3.19}$$

We finally compute the Killing potential for this rescaled matter field Ψ'^A [69]. Using the general definition for Killing potentials (2.17) we find

$$-iM_{m\,i} = i\,qf^2\, M_{\sigma\,i} K_m + \bar{\Psi}'^{\underline{A}} e^{-qf^2 K_\sigma} R_{i\,\underline{A},A}\Psi'^A, \tag{3.20}$$

with $R_{i\underline{A},A} = G_{\underline{A}B} R^B_{i,A} + G_{\underline{A}A,\beta} R^\beta_i$. The Killing potential $M_{\sigma\,i}$ is given by the usual expression (2.17).

3.4 Bundles over S^2 and H^2

We illustrate matter coupling to a supersymmetric σ-model and bundles over Kähler manifolds by returning to the two examples of the sphere and hyperbolic space discussed in subsections 2.3.1 and 2.2.1. We first discuss different forms of matter coupling and after that show that S^2 is a non-trivial Kähler-Hodge manifold by computing the integral of the Kähler form over the sphere itself.

We discuss matter coupling to S^2 and H^2 at the same time, using $\eta = \pm 1$ to distinguish both manifolds. We take the matter multiplet $\Psi = (x, \phi_L, h)$ to be a contravariant vector instead of a tangent vector, since we use this matter representation later. Its scalar component transforms as the derivative $\frac{\partial}{\partial z}$ under the isometries

$$\delta(\theta)x = -2\,(\eta f z\bar{\epsilon} + i\theta_Y)\,x, \tag{3.21}$$

using a similar transformation as is given in eqs. (3.10) and (2.49). As H^2 is covered with just one coordinate patch, we only give the transition function for the change of coordinates $z_- \to z_+$ on the sphere S^2. From eq. (2.32) we infer that the sections x_+ and x_- on the different coordinate patches relate to each other by

$$x_+ = -\left(\frac{z_-}{2R}\right)^2 x_-, \tag{3.22}$$

for all $z_-, z_+ \neq 0$. The simplest invariant matter Kähler potential is

$$K_m(\bar{z}, \bar{x}; z, x) = \bar{x} G_\sigma^{-1} x = (1 + \eta f^2 \bar{z} z)^2 \bar{x} x. \tag{3.23}$$

The non-trivial singlet as defined in eq. (3.14) is the other important example of matter coupling. We obtain the transformation rules for the scalar s of the chiral multiplet $\Omega = (s, \chi_L, \mathrm{f})$

$$\delta(\theta) s = q\, \eta f^2 F(z; \theta) s = q\, (\eta f z \bar{\epsilon} + i\theta_Y)\, s, \tag{3.24}$$

using eq. (2.53) with $q \in \mathbb{R}$. The invariant Kähler potential for s is given by

$$\bar{s} e^{q\, \eta f^2 K_\sigma} s = \frac{\bar{s} s}{(1 + \eta f^2 \bar{z} z)^q}. \tag{3.25}$$

The transformation (2.38) of the Kähler potential K_{S^2} when changing the coordinate system results in a transition of s

$$s_+ = \left(i \frac{2R}{z_-}\right)^q s_-, \quad s_- = \left(-i \frac{2R}{z_+}\right)^q s_+ \tag{3.26}$$

by eq. (B.3). Notice that if we can take $q \notin \mathbb{Z}$, we introduce a logarithm that is not holomorphic; this is a heuristic derivation of the line bundle quantization. When we take $q = -2$ we obtain the same transformation rule as for the contravariant vector x. This is not a coincidence as the (co)tangent space is one-dimensional, hence all matter multiplets are one-dimensional representations of $U(1)$. (The tangent bundle and line bundles of higher dimensional Kähler manifolds are not so alike.)

For S^2 we now compute the integral of the Kähler form of the Kähler potential $qf^2\, K_\sigma$ corresponding to the line bundle discussed above

$$\omega(qf^2\, K_\sigma) = -iqf^2\, \frac{d\bar{z} \wedge dz}{(1 + f^2 \bar{z} z)^2} \tag{3.27}$$

over the smallest two-cycle, S^2 itself. When making a change of coordinates $fz = re^{i\phi}$ we find

$$\int_{S^2} \omega = q \int_0^{2\pi} d\phi \int_0^\infty \frac{2r dr}{(1 + r^2)^2} = 2\pi\, q. \tag{3.28}$$

Hence we find again that $q \in \mathbb{Z}$. A section of the minimal line bundle is obtained when $q = \pm 1$.

3.5 Supergravity

We discuss the coupling of σ-model and matter multiplets to supergravity with possible gaugings of non-linear symmetries. This discussion reviews and generalizes the work by Cremmer et al. [55, 77, 78] and Kugo et al. [56] to include non-linear gauge transformations. The coupling of a supersymmetric non-linear σ-model to supergravity has also been discussed in [79] using the superspace formalism [14]. A related approach using Kähler superspace [80] can be found in [81]. A modern overview of the various approaches can be found in ref. [82].

The main purpose of this section is to show that the rescaling of matter multiplets we discussed in section 3.3 can arise naturally in supergravity. We first review the coupling of chiral multiplets to supergravity using superconformal tensor calculus in general. After that we focus on the peculiarities when these chiral multiplets are σ-model and matter superfields.

3.5.1 Gauged Fixed Superconformal Gravity

As was discussed in ref. [55, 83, 84] an elegant way of coupling chiral multiplets to supergravity is performed as follows: first couple the chiral multiplets $\Sigma^{\mathcal{A}} = (Z^{\mathcal{A}}, \psi_L^{\mathcal{A}})$ and vector multiplets $V^i = (A_\mu^i, \lambda^i)$ to superconformal gravity, using a compensating chiral multiplet Ω. By fixing a set of gauges involving this compensating chiral multiplet Ω the superconformal algebra is reduced to the super-Poincaré algebra.

The super-Poincaré algebra consists of the supersymmetry spinor generator Q, the Lorentz transformation and the rotation generators M_{mn} and the translation generators P_n. This algebra is extended to the super-conformal algebra, that also includes an additional supersymmetry generator S, the special conformal boost generators K_m, the chiral rotation generator A and finally the dilatation generator D. The commutation relations of these generators can be found in refs. [83, 84]. On any generic chiral multiplet $\Sigma = (Z, \psi_L, H)$ the local superconformal algebra with transformations $\delta = \delta_Q(\epsilon) + \delta_S(\eta) + \delta_D(\lambda) + \delta_A(\theta)$ is realized by

$$\delta Z = \bar{\epsilon}_R \psi_L + \omega\big(\lambda - \tfrac{i}{3}\,\theta\big) Z \tag{3.29}$$

$$\delta\psi_L = \frac{1}{2}\,(\not{D} Z \epsilon_R + H \epsilon_L) + \omega Z \eta_L + \Big[\big(\omega + \tfrac{1}{2}\big)\lambda + i\big(\tfrac{1}{2} - \tfrac{\omega}{3}\big)\theta\Big]\psi_L,$$

$$\delta H = \bar{\epsilon}_L\left(\not{D}\psi_L - \lambda_R^i \mathcal{R}_i(Z)\right) + 2(1-\omega)\bar{\eta}_R \psi_L + \Big[(\omega+1)\lambda + i\big(1 - \tfrac{\omega}{3}\big)\theta\Big] H.$$

Here $\lambda,\ \theta$ are the parameters of local scale (D) and chiral $U_A(1)$ transformations and the spinors $\epsilon,\ \eta$ parameterize local Q- and S-supersymmetry transformations, respectively. Furthermore $\omega^{(\mathcal{A})}$ denotes the Weyl weight of the chiral multiplet $\Sigma^{\mathcal{A}}$; the Weyl weight of Ω is taken to be $\omega^{(\Omega)} = 1$. The special conformal boosts do not have to be considered here as their only role is to fix the Weyl gauge field b_μ

to zero when we restrict to Poincaré supergravity, as we will see in eq. (3.34). The covariant derivatives are superconformal derivatives with the non-linear gauge-covariantizations (2.42) included. It is convenient to rescale the chiral multiplets $\Sigma^{\mathcal{A}}$ by multiplying with an appropriate power of the compensator Ω such that their Weyl weights vanish:

$$\Sigma'^{\mathcal{A}} = \Omega^{-\omega(\mathcal{A})}\Sigma^{\mathcal{A}} \quad \Rightarrow \quad \omega'^{(\mathcal{A})} = 0. \tag{3.30}$$

Clearly, the dimension of the physical fields (as opposed to the Weyl weight) is kept fixed by taking Ω dimensionless.

Locally superconformally invariant Lagrangeans cannot be obtained by simply taking the F- and D-terms of (composite) chiral and vector multiplets respectively. However there exist generalizations of the F- and D-terms that are superconformally invariant, called density formulae [56, 83, 84]. Since the superconformal group also contains Weyl rescalings, the action in $D = 4$ is only invariant under these rescalings if the Lagrangean has Weyl weight 4. The Weyl weights of different components are not the same. In fact from (3.29) we see that an arbitrary chiral multiplet with Weyl weight ω has an F-term with Weyl weight $\omega + 1$. Therefore only the F-term density formula, denoted $[\Psi]_F$, for a chiral multiplet Ψ of Weyl weight 3 gives rise to an invariant action. Similarly a vector multiplet V of Weyl weight 2 gives an invariant action using a D-term density formula $[V]_D$.

The most general superconformally invariant Lagrangean that contains at most two spacetime derivatives [79, 80] is given by

$$\mathcal{L} = \frac{1}{\kappa^2}\left[3\,\bar{\Omega}\Omega e^{-\frac{1}{3}\kappa^2 K(\bar{\Sigma}',\Sigma')}\right]_D + \frac{1}{\kappa^3}\,[\Omega^3\mathcal{W}(\Sigma')]_F + [f_{ij}W^i(V)W^j(V)]_F. \tag{3.31}$$

Here $\kappa^2 = 1/m_P^2 = 8\pi/M_P^2 = 8\pi G_N$ is the inverse reduced Planck mass squared and G_N is Newton's constant. As the chiral multiplets $\Sigma'^{\mathcal{A}}$ do not have a Weyl weight, the compensator Ω is the only chiral multiplet that can be used to construct superfields with the appropriate Weyl weights that result in Weyl rescaling invariant actions. By redefining the compensating multiplet as

$$\Omega' = \mathcal{W}^{1/3}(\Sigma')\,\Omega, \tag{3.32}$$

this Lagrangean can be cast in the form [55, 56]

$$\mathcal{L} = \frac{1}{\kappa^2}\left[3\bar{\Omega}'\Omega' e^{-\frac{1}{3}\kappa^2\mathcal{K}}\right]_D + \frac{1}{\kappa^3}\,[\Omega'^3]_F + [f_{ij}W^iW^j]_F, \tag{3.33}$$

where $\mathcal{K}$ is given by eq. (3.8). To reduce the Lagrangean (3.33) to Poincaré supergravity with matter coupled to it, one has to perform a number of gauge-fixings [55]. This can be done in a clever way [56], which avoids Weyl rescaling and chiral rotations, by choosing

$$\begin{aligned} D\ &: \quad 3\bar{s}'s'e^{-\frac{1}{3}\kappa^2\mathcal{K}} = 3, \qquad & A\ &: \quad \mathrm{Im}\ s' = 0, \\ S\ &: \quad \chi'_L = -\kappa^2 s'\mathcal{K}_{,\mathcal{A}}\psi_L^{\mathcal{A}}, \qquad & K_m &: \quad b_\mu = 0, \end{aligned} \tag{3.34}$$

using the components of $\Omega' = (s', \chi'_L)$. The factors 3 and $-\frac{1}{3}$ in the D-term in eqs. (3.31) and (3.33) are chosen such that this reduction gives the standard normalizations of the (super)gravity and the kinetic terms of the chiral multiplets. Indeed, only if we take $e^H = 3e^{-\frac{1}{3}\kappa^2 K}$ the expansion of the D term

$$\frac{1}{\kappa^2}\left[\bar{\Omega}'\Omega' e^H\right]_D = \frac{1}{\kappa^2}\, e\, \bar{s}' s' e^H \left(-\tfrac{1}{6}R + H_{\underline{\mathcal{A}}\mathcal{A}} D_\mu \bar{Z}^{\underline{\mathcal{A}}} D^\mu Z^{\mathcal{A}}\right) + \dots\,, \tag{3.35}$$

and the standard form is obtained. Here e is the determinant of the spacetime vielbein and the dots indicate that we have neglected a lot of other terms, that are irrelevant for the argument above.

The gauge fixings eq. (3.34) with Ω instead of Ω' for the Lagrangean in the original form (3.31) are not invariant under the Kähler transformations (2.7)

$$K \to K + F + \bar{F}, \quad \Omega \to e^{\frac{1}{3}\kappa^2 F}\Omega, \quad \text{and} \quad \mathcal{W} \to e^{-\kappa^2 F}\mathcal{W}. \tag{3.36}$$

This can be compensated by a chiral rotation [79]

$$\psi = e^{i\frac{1}{2}\kappa^2 \mathrm{Im}(F)\,\gamma_5}\psi \tag{3.37}$$

on all spinors ψ. This is the basis of Kähler superspace [80] where the Kähler $U(1)$ transformations are gauged. Notice that the redefinition of the compensator (3.32) is a special case of this. It is clear that the Lagrangean (3.31) is invariant under these transformations. This reflects the fact that in supergravity the Kähler potential K and the superpotential $\mathcal{W}$ are not independent.

3.5.2 σ-Model and Matter Multiplets Coupled to Supergravity

We make the same distinction between σ-model multiplets Φ^α and matter multiplets Ψ^A as in section 3.2 with the covariant Kähler potential $K = K_\sigma + K_m$ of the combined σ-model and matter system. The coupling of chiral σ-model and matter multiplets to superconformal gravity is performed in an analogous fashion as discussed above, therefore we focus on the special features due to non-linear isometries and global aspects. The coupling of pure σ-models (coset models) to supergravity has been considered in ref. [85].

Under the internal symmetries the σ-model fields Φ^α and the matter fields Ψ^A transform according to eqs. (3.1). Generically this requires the conformal Weyl weights of the σ-model bosons z^α to vanish [68]. This can be derived by requiring the internal symmetries and the space-time symmetries to commute:

$$0 = [\delta_D, \delta_i] z^\alpha = \omega^{(\beta)} R^\alpha_{i,\beta} z^\beta - \omega^{(\alpha)} R^\alpha_i \quad \Rightarrow \quad \omega^{(\alpha)} = 0, \forall\alpha. \tag{3.38}$$

Furthermore, the Weyl weights $\omega^{(A)}$ of the matter multiplets Ψ^A in a single irreducible representation must all be equal; this follows from a similar argument as above for x^A. Using the rescaling (3.30) with the compensator Ω we can make the Weyl weights of the matter multiplets vanish as before, but this changes their transformation rules under internal symmetries. The D-term in Lagrangean (3.31) is only invariant when the compensating superfield Ω transforms as a non-trivial singlet (3.14) under the internal symmetries as

$$\delta_i \Omega = \tfrac{1}{3}\kappa^2 F_i(\Phi)\Omega, \tag{3.39}$$

because the σ-model Kähler potential transforms covariantly (3.9) with the holomorphic function $F_i(\Phi)$. This implies that the multiplet Ψ'^A transforms under the internal symmetries as

$$\delta_i \Psi'^A = R^A_{i\,B}(\Phi)\Psi'^B - \tfrac{1}{3}\omega^{(A)}\kappa^2 F_i(\Phi)\Psi'^A, \tag{3.40}$$

which is precisely the form of equation (3.18) with $q^{(A)} f^2 \to -\frac{1}{3}\omega^{(A)}\kappa^2$. The covariance (3.39) of the compensator Ω also implies that the superpotential $\mathcal{W}$ transforms covariantly

$$\delta_i \mathcal{W}(\Phi,\Psi) = -\kappa^2\, F_i(\Phi)\mathcal{W}(\Phi,\Psi), \tag{3.41}$$

so that the Lagrangean (3.31) is invariant. Comparing this with (3.7), we see that $\kappa^2 = -f^2\, q$.

Although we now consider non-linear internal symmetries the Kähler potential $\mathcal{K}$ takes the same form as given in ref. [55]. But because of this non-linear nature, the gauging of part of the internal symmetries leads to some modifications of the invariant Lagrangean. All gauge couplings involve the Killing vectors $(T_i Z)^{\mathcal{A}} \to \mathcal{R}_i^{\mathcal{A}}$ now. The D-terms and the gaugino-matter couplings in eq. (3.14) of ref. [55] are

$$\begin{aligned} e^{-1}\,\Delta\mathcal{L} \;=\; & \tfrac{1}{4} f_{ij}\, D^i D^j + \tfrac{1}{2\kappa^2}\left(e^{\mathcal{G}}\right)_{,\mathcal{A}}(T_i Z)^{\mathcal{A}}\left(D^i + i\bar{\Psi}_R\cdot\gamma\lambda^i_R\right) \\ & -i\tfrac{2}{\kappa^2}\left(e^{\mathcal{G}}\right)_{,\underline{\mathcal{A}}\mathcal{A}}(T_i Z)^{\mathcal{A}}\bar{\lambda}^i_L \chi^{\underline{\mathcal{A}}}_R + \text{h.c.}, \end{aligned} \tag{3.42}$$

with $e^{\mathcal{G}} = 3e^{-\frac{1}{3}\kappa^2 K}\bar{\Omega}\Omega = 3e^{-\frac{1}{3}\kappa^2\mathcal{K}}\bar{\Omega}'\Omega'$. This can be written conveniently in terms of Killing potentials $-i\mathcal{M}_i = \frac{1}{\kappa^2}\left(e^{\mathcal{G}}\right)_{,\mathcal{A}}\mathcal{R}_i^{\mathcal{A}}$, where $\mathcal{R}_i^{\mathcal{A}}$ are the Killing vectors defined for all the fields in the model, as

$$e^{-1}\,\Delta\mathcal{L} \;=\; \frac{1}{4} f_{ij}\, D^i D^j \;+\; \frac{1}{2}\mathcal{M}_i\left(D^i + i\bar{\Psi}_R\cdot\gamma\lambda^i_R\right) - 2i\mathcal{M}_{i,\underline{\mathcal{A}}}\bar{\lambda}^i_L\chi^{\underline{\mathcal{A}}}_R + \text{h.c.}. \tag{3.43}$$

Here we have used that $e^{\mathcal{G}}$ is invariant and hence does not give rise to a holomorphic function $\mathcal{F}_i$. The total Killing potential $\mathcal{M}_i$ for the combined system of σ-model and matter multiplets can be written as $\mathcal{M}_i = e^{\mathcal{G}}\left(M_{\sigma\, i} + M_{m\, i}\right)$ where $M_{\sigma\, i}$ is the σ-model Killing potential and $M_{m\, i}$ the matter Killing potential given in eq. (3.20).

Now we turn to a discussion of some global aspects of σ-models coupled to supergravity. First of all it is not always possible to use the redefinition (3.32) of the compensator Ω to obtain an invariant Kähler potential (3.8). For these transformations to make sense the superpotential should be non-zero and non-singular [56, 72].

When the σ-model multiplets correspond to a topologically non-trivial manifold, one has different Kähler potentials $K_{(a)}$ defined on each coordinate patch separately. On the overlap two of them may differ by a sum of a holomorphic function $F_{(ab)}$ and its conjugate. For globally supersymmetric Lagrangeans this holomorphic function is irrelevant as the Lagrangean only contains the metric, the second mixed derivative of the Kähler potentials. We assume here that we have chosen the Kähler potential K such that it gives the minimal kinetic terms for the scalars when the σ-model scale is taken to infinity. This is equivalent to the demand that integrating the corresponding Kähler form over a generating 2-cycle, it gives 2π. (This can be seen by restricting the Kähler potential K to the embedding $\mathbb{C}P^1$ that wraps once. This gives the $\mathbb{C}P^1$ Kähler potential which is canonically normalized and the integral over the Kähler form is equal to 2π.) As remarked by Witten and Bagger [86] in supergravity the chiral rotations (3.37) of the fermions are only globally well defined when eq. (B.5) of appendix B with $f \to \kappa$ is satisfied. Hence only Kähler-Hodge manifolds can be coupled to supergravity and Newton's constant is quantized in units of the σ-model scale: $\frac{\kappa^2}{f^2} = -q \in \mathbb{Z}$. Finally from (3.36) it can be inferred [86] that the covariant superpotential is a section of a line bundle. Notice that this implies that $f^2 \leq \kappa^2$.

We conclude with a discussion of the scalar potential that arises in supergravity [55, 87]. After eliminating the auxiliary fields D^i and $H^{\mathcal{A}}$ the scalar potential is given by

$$V = \frac{1}{2}[(\mathrm{Re} f)^{-1}]^{ij}\mathcal{M}_i\mathcal{M}_j + \left(G^{\mathcal{A}\underline{\mathcal{A}}}\mathcal{K}_{,\mathcal{A}}\mathcal{K}_{,\underline{\mathcal{A}}} - 3\kappa^2|\mathcal{W}|^2\right) e^{\kappa^2 K} \tag{3.44}$$

with $\mathcal{K}_{,\mathcal{A}} = \kappa^2\mathcal{W}K_{,\mathcal{A}} + \mathcal{W}_{,\mathcal{A}} = qf^2\mathcal{W}K_{,\mathcal{A}} + \mathcal{W}_{,\mathcal{A}}$. We denote by $\langle A\rangle$ the vacuum expectation value of a physical quantity A. This potential is not automatically non-negative; necessary and sufficient conditions for a non-negative potential have been given in ref. [88].

We have F-, D-term supersymmetry breaking when $\langle\mathcal{K}_{,\mathcal{A}}\rangle \neq 0$ for an index $\mathcal{A}$, $\langle\mathcal{M}_i\rangle \neq 0$ for an index i, respectively. As $\mathcal{W}$ transforms covariantly under the internal symmetries, $\langle\mathcal{W}\rangle \neq 0$ breaks a $U(1)$ isometry. When on the other hand $\langle\mathcal{W}\rangle = 0$ the gravitino mass vanishes and the potential is non-negative. F-term supersymmetry breaking is in that case equivalent to the globally supersymmetric condition that $\langle\mathcal{W}_{,\mathcal{A}}\rangle \neq 0$. We study the consequences of these observation for the Grassmannian standard models in chapter 5.

3.6 Block-Diagonal Metric, Covariant Fermions

If one considers the combined system of σ-model and matter multiplets, the metric of that total system is in general not diagonal: different representations of the symmetry algebra mix in the quadratic kinetic terms of the scalars and chiral fermions. This is carried over to the definition of propagators. If one knows that the theory is constructed out of several sectors, one would like to be able to assign to each sector a separate block in the metric, without mixing between different sectors.

With the method we show, that the metric can be made block-diagonal and the fermion states $\hat{\chi}_L^A$, that are covariant under the internal symmetries can be determined. (In section 3.2 we saw that the fermions of matter multiplets χ_L^A do not transform covariantly under the internal symmetries (3.3).) We also derive the form the Lagrangean for the fermions takes, when written in terms of the fermions $\hat{\chi}_L^A$. The method we follow generalizes the result of ref. [89] where only quadratically coupled rank-1 matter was considered: $K_m = \bar{x}^{\underline{\alpha}} G_{\sigma\,\underline{\alpha}\alpha} x^\alpha$, with the metric $G_{\sigma\,\underline{\alpha}\alpha}$ depending on z^α and $\bar{z}^\alpha$. With the machinery of non-holomorphic transformations developed in appendix C this can be done elegantly without too much computational difficulty.

We consider a Kähler manifold which is parameterized by the coordinates $Z^{\mathcal{A}} = (z^\alpha, x^A)$ and their conjugates with Kähler potential K. First we identify the σ-model Kähler potential $K_\sigma(\bar{z}, z)$ and the matter Kähler potential $K_m(\bar{x}, x; \bar{z}, z)$ by

$$K_\sigma = K\big|_{x=\bar{x}=0}\,, \qquad K_m \equiv G_{\bar{x}x} = K - K_\sigma. \tag{3.45}$$

The notation $G_{\bar{x}x}$ for the matter Kähler potential is very suggestive as it reduces to $G_{\bar{x}x} = \bar{x}^{\underline{A}} G_{\underline{A}A} x^A$ if matter is quadratically coupled. To take this analogy with the case of quadratic matter coupling a bit further, we define

$$G_{\bar{x}A} = K_{,A}, \qquad G_{\underline{A}x} = K_{,\underline{A}}, \tag{3.46}$$

while the metrics for the matter and σ-model fields are

$$G_{\underline{A}A} \equiv K_{,\underline{A}A}, \qquad G_{\sigma\,\underline{\alpha}\alpha} \equiv K_{\sigma,\underline{\alpha}\alpha}. \tag{3.47}$$

To be able to use the method explained in appendix C, we first need to define the non-holomorphic transformation matrices $X_{\mathcal{A}}^{\mathcal{A}'}$ and $\bar{X}_{\underline{\mathcal{A}}}^{\underline{\mathcal{A}}'}$. We do this by demanding that the transformations (C.5) block-diagonalize the metric of the combined system of σ-model fields but at the same time leave the metric for the matter fields unchanged. The metric of the combined system is

$$G_{\underline{\mathcal{A}}\mathcal{A}} = \begin{pmatrix} G_{\sigma\,\underline{\alpha}\alpha} + G_{\bar{x}x,\underline{\alpha}\alpha} & G_{\bar{x}A,\underline{\alpha}} \\ G_{\underline{A}x,\alpha} & G_{\underline{A}A} \end{pmatrix}, \tag{3.48}$$

where $G_{\sigma\,\underline{\alpha}\alpha}$ is the metric of the σ-model without matter coupling. Since the σ-model submetric $G_{\sigma\,\underline{\alpha}\alpha}$ is already modified by the matter coupling, it is most convenient to put the modifications due to the diagonalization there as well. The appropriate transformation is given by the matrices

$$X^{\mathcal{A}}_{\mathcal{A}'} = \begin{pmatrix} \delta^{\alpha}_{\alpha'} & 0 \\ -\Gamma^{A}_{x\alpha'} & \delta^{A}_{A'} \end{pmatrix}, \quad \bar{X}^{\underline{\mathcal{A}}}_{\underline{\mathcal{A}}'} = \begin{pmatrix} \delta^{\underline{\alpha}}_{\underline{\alpha}'} & -\bar{\Gamma}^{\underline{A}}_{\bar{x}\underline{\alpha}'} \\ 0 & \delta^{\underline{A}}_{\underline{A}'} \end{pmatrix}. \tag{3.49}$$

In analogy to the quadratically coupled case [89, 69] we have introduced generalizations of the connections

$$\begin{aligned} \Gamma^{\alpha}_{\beta\gamma} &\equiv G_{\sigma}^{\alpha\underline{\alpha}} G_{\sigma\,\underline{\alpha}\beta,\gamma}, & \Gamma^{A}_{B\gamma} &\equiv G^{A\underline{A}} G_{\underline{A}B,\gamma}, \\ \Gamma^{A}_{BC} &\equiv G^{A\underline{A}} G_{\underline{A}B,C}, & \Gamma^{A}_{x\gamma} &\equiv G^{A\underline{A}} G_{\underline{A}x,\gamma} \end{aligned} \tag{3.50}$$

and their conjugates. (There is no object Γ^{A}_{xC} since a similar definition as in eqs. (3.50) just gives $\Gamma^{A}_{xC} = \delta^{A}_{C}$.) Indeed, the metric of the full system after this transformation is

$$G_{\underline{\mathcal{A}}'\mathcal{A}'} = \begin{pmatrix} G_{\underline{\alpha}'\alpha'} & 0 \\ 0 & G_{\underline{A}'A'} \end{pmatrix}, \tag{3.51}$$

with the effective metric for the $z^{\alpha}, \bar{z}^{\underline{\alpha}}$ scalars given by

$$G_{\underline{\alpha}\alpha} = G_{\sigma\,\underline{\alpha}\alpha} + R_{\bar{x}x\underline{\alpha}\alpha}. \tag{3.52}$$

In this derivation we have assumed that the metric $G_{\underline{A}A}$ is invertible, and we have used the generalized curvature $R_{\bar{x}x\underline{\alpha}\alpha}$ defined by

$$R_{\bar{x}x\underline{\alpha}\alpha} \equiv g_{\bar{x}x,\underline{\alpha}\alpha} - G_{\bar{x}B,\underline{\alpha}} G^{B\underline{B}} G_{\underline{B}x,\alpha} = G_{\bar{x}x,\underline{\alpha}\alpha} - \bar{\Gamma}^{\underline{B}}_{\bar{x}\underline{\alpha}} G_{\underline{B}B} \Gamma^{B}_{x\alpha}. \tag{3.53}$$

In the following we also assume that the metric (3.51) is invertible. Notice that the inverse of this transformation (3.49) is given by the same matrices but the primed indices now are upstairs and there is an additional minus-sign in front of the off-diagonal parts.

Using the connections (3.50) one can define quite a number of generalized curvature components

$$\begin{aligned} R_{\underline{\alpha}\alpha\underline{\beta}\beta} &\equiv G_{\sigma\,\underline{\alpha}\gamma} \left(\Gamma^{\gamma}_{\alpha\beta}\right)_{,\underline{\beta}} = G_{\sigma\,\underline{\gamma}\alpha} \left(\bar{\Gamma}^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}}\right)_{,\beta}, \\ R_{\underline{A}A\underline{B}B} &\equiv G_{\underline{A}C} \left(\Gamma^{C}_{AB}\right)_{,\underline{B}} = G_{\underline{C}A} \left(\bar{\Gamma}^{\underline{C}}_{\underline{A}\underline{B}}\right)_{,B}, \\ R_{\underline{A}A\underline{\alpha}\alpha} &\equiv G_{\underline{A}C} \left(\Gamma^{C}_{A\alpha}\right)_{,\underline{\alpha}} = G_{\underline{C}A} \left(\bar{\Gamma}^{\underline{C}}_{\underline{A}\underline{\alpha}}\right)_{,\alpha}, \\ R_{\bar{x}A\underline{\alpha}\alpha} &\equiv G_{\underline{C}A} \left(\bar{\Gamma}^{\underline{C}}_{\bar{x}\underline{\alpha}}\right)_{,\alpha}, \qquad R_{\underline{A}x\underline{\alpha}\alpha} \equiv G_{\underline{A}C} \left(\Gamma^{C}_{x\alpha}\right)_{,\underline{\alpha}}, \\ R_{\bar{x}A\underline{\alpha}B} &\equiv G_{\underline{C}A} \left(\bar{\Gamma}^{\underline{C}}_{\bar{x}\underline{\alpha}}\right)_{,B}, \qquad R_{\underline{A}x\underline{B}\alpha} \equiv G_{\underline{A}C} \left(\Gamma^{C}_{x\alpha}\right)_{,\underline{B}}. \end{aligned} \tag{3.54}$$

Other generalized curvature components either vanish or are irrelevant in the following.

If the transformations described by the matrices (3.49) are applied to the derivative of the coordinates z^α, x^A we find that (C.4)

$$(\partial_\mu Z)'^{\mathcal{A}'} = \begin{pmatrix} \partial_\mu z^{\alpha'} \\ D_\mu x^{A'} \end{pmatrix} \equiv \begin{pmatrix} \partial_\mu z^{\alpha'} \\ \partial_\mu x^{A'} + \Gamma^{A'}_{x\beta'} \partial_\mu z^{\beta'} \end{pmatrix}. \tag{3.55}$$

The derivative $D_\mu x^{A'}$ is covariant under holomorphic transformations. From now on we drop the primes on the indices if no confusion is possible. Using these definitions the kinetic energy of the boson fields z^α and x^A can be written as

$$-\mathcal{L}_B = G_{\underline{\alpha}\alpha} \partial^\mu \bar{z}^{\underline{\alpha}} \partial_\mu z^\alpha + G_{\underline{A}A} D^\mu \bar{x}^{\underline{A}} D_\mu x^A. \tag{3.56}$$

So far we have only discussed how the metric and covariant vectors behave under the transformations described by the matrices (3.49). This is sufficient to write the kinetic Lagrangean for the complex scalars in a convenient form. We now turn to the calculation of the kinetic Lagrangean for the chiral fermions. First of all we have to discuss fermions themselves. As fermions $\psi_L^{\mathcal{A}} = (\psi_L^\alpha, \chi_L^A)$ are covariant objects on the manifold, they can also be rewritten using transformation (3.49)

$$\psi_L^{\mathcal{A}'} = \begin{pmatrix} \psi_L^{\alpha'} \\ \chi'^{A'}_L \end{pmatrix} = \begin{pmatrix} \psi_L^{\alpha'} \\ \chi_L^{A'} + \Gamma^{A'}_{x\beta} \psi_L^\beta \end{pmatrix}. \tag{3.57}$$

The fermion χ_L^A is turned into a covariant vector $\hat{\chi}_L^A$ by this transformation [73]

$$\chi'^A_L = \chi_L^A + \Gamma^A_{x\beta} \psi_L^\beta \equiv \hat{\chi}_L^A \tag{3.58}$$

where the hat denotes covariantization, as can be checked explicitly using the transformation properties (3.3).

The kinetic terms of the fermions in eq. (2.39) involve the covariant derivative on the chiral fermions of the full system, so we have to know the form of the covariant derivate on a covariant vector $V^{\mathcal{A}}$. To calculate this we use eq. (C.8):

$$(\mathcal{D}_\mu V)_{\underline{A}'} = (\mathcal{D}_\mu V)'_{\underline{A}'} - \bar{U}^{\underline{\mathcal{E}}'}_{\underline{A}'\mathcal{E}'} G_{\underline{\mathcal{E}}'\mathcal{B}'} \partial_\mu z^{\mathcal{E}'} V^{\mathcal{B}'} + G_{\underline{A}'\mathcal{E}'} U^{\mathcal{E}'}_{\mathcal{B}'\underline{\mathcal{E}}'} \partial_\mu \bar{z}^{\underline{\mathcal{E}}'} V^{\mathcal{B}'}. \tag{3.59}$$

This means we have to calculate the non-vanishing contributions to the connection $\Gamma'^{\mathcal{A}'}_{\mathcal{B}'\mathcal{C}'} = G'^{\mathcal{A}'\underline{\mathcal{A}}'} G'_{\underline{\mathcal{A}}'\mathcal{B}',\mathcal{C}'}$ of the full system

$$\begin{aligned} G'_{\underline{\alpha}'\alpha',\mathcal{C}'} &= \left(G_{\underline{\alpha}'\delta'} \Gamma^{\delta'}_{\alpha'\gamma'} + R_{\bar{x}x\underline{\alpha}'\alpha';\gamma'} + R_{\bar{x}B\underline{\alpha}'C} \Gamma^B_{x\alpha'} \Gamma^C_{x\gamma'}, \quad R_{\bar{x}x\underline{\alpha}'\alpha',C'} \right) \\ G'_{\underline{A}'A',\mathcal{C}'} &= G_{\underline{A}'B'} \left(\hat{\Gamma}^{B'}_{A'\gamma'}, \quad \Gamma^{B'}_{A'C'} \right) \equiv \left(G_{\underline{A}'B'} \left(\Gamma^{B'}_{x\gamma'} \right)_{,A'}, \quad G_{\underline{A}'B'} \Gamma^{B'}_{A'C'} \right) \end{aligned} \tag{3.60}$$

which involves the metric $G_{\underline{\alpha}'\delta'}$ of the transformed system. On the r.h.s. the index $\mathcal{C}' = (\gamma', C')$ is written out explicitly using a row-vector notation. The non-vanishing components of $U^{\mathcal{A}'}_{\mathcal{B}'\underline{\mathcal{C}}'}$ are:

$$U^{A'}_{\beta'\underline{\mathcal{C}}'} = -G^{A'\underline{A}'} \begin{pmatrix} R_{\underline{A}'x\underline{\gamma}'\beta'} - R_{\underline{A}'x\underline{D}'\beta'}\bar{\Gamma}^{\underline{D}'}_{\bar{x}\underline{\gamma}'} \\ R_{\underline{A}'x\underline{C}'\beta'} \end{pmatrix}. \tag{3.61}$$

In these expressions we have made use of a covariant derivative in $R_{\bar{x}x\underline{\alpha}'\alpha';\gamma'}$, which is defined in the usual way using the connections given in equations (3.50), and we have used the identities

$$\left(\Gamma^A_{x\gamma}\right)_{,C} = \Gamma^A_{C\gamma} - \Gamma^A_{BC}\Gamma^B_{x\gamma} \equiv \hat{\Gamma}^A_{C\gamma}, \quad R_{\bar{x}x\underline{\alpha}\alpha,B} = R_{\bar{x}B\underline{\alpha}\alpha} - R_{\bar{x}C\underline{\alpha}B}\Gamma^C_{x\gamma}. \tag{3.62}$$

With this it is easy to give the rewritten covariant derivative explicitly. As an application we give here the kinetic terms of the supersymmetric Lagrangean (2.39) for the chiral fermions including covariantizations:

$$\begin{aligned} -\mathcal{L}_F = {} & G_{\underline{\alpha}'\alpha'}\bar{\psi}^{\underline{\alpha}'}_L \overset{\leftrightarrow}{\not{D}} \psi^{\alpha'}_L + G_{\underline{A}'A'}\hat{\bar{\chi}}^{\underline{A}'}_L \overset{\leftrightarrow}{\not{D}} \hat{\chi}^{A'}_L \\ & + \left(R_{\bar{x}x\underline{\alpha}'\alpha';\gamma'} + R_{\bar{x}B'\underline{\alpha}'C'}\Gamma^{B'}_{x\alpha'}\Gamma^{C'}_{x\gamma'}\right)\partial_\mu z^{\alpha'}\bar{\psi}^{\underline{\alpha}'}_L\gamma^\mu\psi^{\gamma'}_L \\ & - \left(R_{\bar{x}x\underline{\alpha}'\alpha';\underline{\gamma}'} + R_{\underline{B}'x\underline{C}'\alpha'}\bar{\Gamma}^{\underline{B}'}_{\bar{x}\underline{\alpha}'}\bar{\Gamma}^{\underline{C}'}_{\bar{x}\underline{\gamma}'}\right)\partial_\mu\bar{z}^{\underline{\alpha}'}\bar{\psi}^{\underline{\gamma}'}_L\gamma^\mu\psi^{\alpha'}_L \\ & - \left[\left(R_{\bar{x}x\underline{\gamma}'\alpha',\underline{A}'} + 2R_{\underline{A}'x\underline{\gamma}'\alpha'} - 2R_{\underline{A}'x\underline{D}'\alpha'}\bar{\Gamma}^{\underline{D}'}_{\bar{x}\underline{\gamma}'}\right)\partial_\mu\bar{z}^{\underline{\gamma}'} + 2R_{\underline{A}'x\underline{C}'\alpha'}D_\mu\bar{x}^{\underline{C}'}\right]\hat{\bar{\chi}}^{\underline{A}'}_L\gamma^\mu\psi^{\alpha'}_L \\ & + \left[\left(R_{\bar{x}x\underline{\alpha}'\gamma',A'} + 2R_{\bar{x}A'\underline{\alpha}'\gamma'} - 2R_{\bar{x}A'\underline{\alpha}'D'}\Gamma^{D'}_{x\gamma'}\right)\partial_\mu z^{\gamma'} + 2R_{\bar{x}A'\underline{\alpha}'C'}D_\mu x^{C'}\right]\bar{\psi}^{\underline{\alpha}'}_L\gamma^\mu\hat{\chi}^{A'}_L \end{aligned} \tag{3.63}$$

with the covariant derivatives defined in eq. (3.55) and

$$\begin{aligned} D_\mu\psi^{\alpha'}_L \equiv {} & \partial_\mu\psi^{\alpha'}_K + \Gamma^{\alpha'}_{\beta'\gamma'}\partial_\mu z^{\gamma'}\psi^{\beta'}_L \\ D_\mu\hat{\chi}^{A'}_L \equiv {} & \partial_\mu\hat{\chi}^{A'}_L + \hat{\Gamma}^{A'}_{B'\gamma'}\partial_\mu z^{\gamma'}\hat{\chi}^{B'}_L + \Gamma^{A'}_{B'C'}\partial_\mu x^{C'}\hat{\chi}^{B'}_L. \end{aligned} \tag{3.64}$$

The four-fermion terms can be calculated by using eq. (C.11) of appendix C.

Let us mention one further important application of the transformation diagonalizing the metric to eq. (3.51). For several physical applications, like determining whether there is soft supersymmetry breaking, one needs to know the contracted connection $\Gamma_{\mathcal{A}} = \Gamma^{\mathcal{B}}_{\mathcal{B}\mathcal{A}}$ and the Ricci-tensor $R_{\underline{\mathcal{A}}\mathcal{A}} = G^{\mathcal{B}\underline{\mathcal{B}}}R_{\underline{\mathcal{B}}\mathcal{B}\underline{\mathcal{A}}\mathcal{A}}$ of the full model. In particular the calculation of the curvature can be very tedious even in the setup presented here, and it is hard to obtain the Ricci tensor in this way. However it is well known that the contracted connection and the Ricci tensor can be obtained from the determinant $\det G$ of the metric, see eq. (2.4). As the transformation matrices (3.49) are upper- or lower-triangular matrices with identities on the diagonal, their determinants are unity. Therefore we may use the block-diagonal metric (3.51) to calculate the determinant of the full metric: $\det G' = \det G$.

Chapter 4

Coset Spaces and Matter Coupling

4.1 Introduction

The focus of this section is on homogeneous coset spaces that are Kählerian and can therefore be used in supersymmetric model building. They provide a basis for the construction of a large set of interesting examples of supersymmetric non-linear σ-models. In later chapters we study several anomaly-free models based on coset spaces in more detail. The present chapter lays the foundation for those discussions.

It is possible to extend a non-Kählerian coset to a Kähler manifold by including so-called quasi-Goldstone bosons [29, 30, 90]. These type of models can arise from spontaneous symmetry within a linear supersymmetric σ-model [47, 53, 91]. More details on the phenomenology of such models can be found in ref. [92, 93]. A theorem of Lerche [94] and Shore [95] states that in such model a part of the linear subgroup is always broken. In other words Kählerian cosets can never arise from spontaneous symmetry breaking. We do not assume that the non-linear σ-model is the effective theory description of a spontaneously broken linear supersymmetric theory. We confine ourselves to Kählerian cosets only.

As we have seen in chapter 2 a Kähler potential determines the Lagrangean of a supersymmetric model. Therefore it is crucial to determine Kähler potentials for Kählerian coset spaces. We discuss three methods to obtain them: the so-called BKMU construction, a non-linear realization of $SL(M+N;\mathbb{C})$ and a Noether procedure for so-called symmetric cosets. In addition we obtain a classification of possible matter couplings to Kählerian cosets.

The most general method to construct a Kähler potential for a Kählerian coset space has been developed by M. Bando, T. Kuramoto, T. Maskawa and S. Uehara (BKMU) [96, 97], which we review first. All types of matter coupling to the coset space can be discussed using the same group theoretical language [98].

The action of any group element of G describes a coordinate transformation on a coset space, therefore the requirement that matter representations are sections of well-defined bundles over a coset space can be formulated entirely within group theory. In this way we can obtain more types of matter representations than one would expect using sections of tensor products of the tangent bundle alone, which were discussed in chapter 3. The Kähler potential associated with a minimal line bundle can be identified explicitly. This results in a classification of all line bundles over Kählerian cosets.

For the second method we study the non-linear action of the special linear group $SL(M+N;\mathbb{C})$ on $M \times N$ complex matrices. By insisting that a Kähler potential is covariant under certain these symmetries, the $SL(M+N;\mathbb{C})$ transformations have to be restricted to at least $SU_\eta(M,N)$ with $\eta = \pm$ distinguishing between the compact group $SU(M+N)$ ($\eta = 1$) and the non-compact group $SU(M,N)$ ($\eta = -1$). Although this method is to a large extent an application of the general method of constructing Kähler potentials, it has many useful features. In particular it shows why the description of the coset spaces $SU_\eta(M,N)/S[U(M) \times U(N)]$, $USp_\eta(N,N)/U(N)$ and $SO(2N)/U(N)$ are quite alike. We also construct elementary matter couplings to these cosets.

The third method introduces complex scalars in the adjoint representation of the algebra of G. For simplicity we only apply this method to so-called symmetric coset spaces. By brute force all scalars are set to constant values that respect the symmetries of the linear subgroup H, except those which are used to parameterize the coset. The transformation rules of the scalars are modified such they still form a representation of G. Using a Noether procedure a power series expression for Kähler potentials is obtained. In this context we discuss the coupling of covariant vectors and non-trivial singlets to the manifold.

In this chapter the focus is on the construction of Kähler potentials for Kählerian coset spaces. To construct the Lagrangean (2.39) explicitly given a Kähler potential is still quite complicated. Using methods described in [99], it can be shown [100, 101, 102] that the Lagrangean (2.39) can be written in terms of Killing vectors (2.9) only.

We briefly illustrate the various methods by revisiting the two-sphere and the hyperbolic space.

4.2 Elements of Lie Group Theory

We develop the notation to discuss a Kählerian homogeneous coset space G/H, where H is a subgroup of a simple Lie group G. The necessary background in Lie group theory can be found in refs. [40, 41, 42]. The non-compactness of G is encoded in the diagonal matrix $\mathfrak{J}$ having only entries ± 1; if all entries are positive G is compact else G is non-compact. The ordering of $\mathfrak{J}$ is chosen such that the top half of $\mathfrak{J}$ has positive entries. The elements g of the group G are chosen to be unitary with respect to $\mathfrak{J}$

$$g^\dagger \mathfrak{J} g = \mathfrak{J} \quad \text{and} \quad \mathbf{a}^\dagger = \mathfrak{J}\mathbf{a}\mathfrak{J}, \tag{4.1}$$

where $g = \exp(i\mathbf{a})$ with $\mathbf{a}$ an element of the algebra of G. According to a theorem by Borel [103] a coset G/H is Kählerian if H is the centralizer $\mathrm{cen}(Y)$ of a torus in G. We assume that H is always a compact subgroup of G; therefore any element of $h \in \mathrm{H}$ commutes with $\mathfrak{J}$: $\mathfrak{J}h = h\mathfrak{J}$. An arbitrary linear combination of $\boldsymbol{\tau}_i$ is denoted by $\boldsymbol{\tau}$. Let $\boldsymbol{\epsilon}_{\boldsymbol{\alpha}}$ be the generator associated with root $\boldsymbol{\alpha}$ and let $\boldsymbol{\epsilon}_{\pm i} = \boldsymbol{\epsilon}_{\pm\boldsymbol{\alpha}_i}$ denote the creation and annihilation operators corresponding to the simple root $\boldsymbol{\alpha}_i$. The generators are taken to be $\mathfrak{J}$-Hermitean $\boldsymbol{\tau}_i^\dagger = \boldsymbol{\tau}_i$ and $\boldsymbol{\epsilon}_{\boldsymbol{\alpha}}^\dagger = \mathfrak{J}\boldsymbol{\epsilon}_{-\boldsymbol{\alpha}}\mathfrak{J}$. The algebra of G can be stated as

$$\begin{gathered}
[\boldsymbol{\tau}_i, \boldsymbol{\tau}_j] = 0, \qquad [\boldsymbol{\epsilon}_i, \boldsymbol{\epsilon}_{-j}] = \delta_{ij}\boldsymbol{\tau}_j, \\
[\boldsymbol{\tau}, \boldsymbol{\epsilon}_{\boldsymbol{\alpha}}] = \boldsymbol{\alpha}(\boldsymbol{\tau})\boldsymbol{\epsilon}_{\boldsymbol{\alpha}}, \qquad [\boldsymbol{\tau}_i, \boldsymbol{\epsilon}_{\pm j}] = \pm G_{ji}\boldsymbol{\epsilon}_{\pm j}, \\
[\boldsymbol{\epsilon}_{\boldsymbol{\alpha}}, \boldsymbol{\epsilon}_{\boldsymbol{\beta}}] = \begin{cases} N_{\boldsymbol{\alpha},\boldsymbol{\beta}}\boldsymbol{\epsilon}_{\boldsymbol{\alpha}+\boldsymbol{\beta}} & \boldsymbol{\alpha}+\boldsymbol{\beta} \text{ a root,} \\ 0 & \text{otherwise.} \end{cases}
\end{gathered} \tag{4.2}$$

The normalization factor $N_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is irrelevant in the following. In this normalization of the algebra the Cartan matrix is $G_{ij} \equiv \frac{2\langle \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j \rangle}{\langle \boldsymbol{\alpha}_j, \boldsymbol{\alpha}_j \rangle} = \boldsymbol{\alpha}_i(\boldsymbol{\tau}_j)$. For the discussion of the general construction of the Kähler potential for the Kählerian coset it is convenient to divide the generators of G into the following sets [96, 97, 104]. We define the set $S = S^\dagger = \{S_a\}$ by

$$S = \{\boldsymbol{\tau}_i \mid \boldsymbol{\epsilon}_i \in \mathrm{H}\} \cup \{\boldsymbol{\epsilon}_{\boldsymbol{\alpha}} \mid \boldsymbol{\epsilon}_{\boldsymbol{\alpha}} \in \mathrm{H}\} \tag{4.3}$$

and the set Y by

$$Y = \left\{ \mathbf{Y}^I = \boldsymbol{\tau}_i (G^{-1})^{iI} \mid \boldsymbol{\tau}_I \notin \mathrm{H} \right\}. \tag{4.4}$$

The set S generates the semi-simple part of H. All generators of S commute with the elements of Y, since

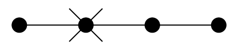
$$[\mathbf{Y}^I, \boldsymbol{\epsilon}_j] = \delta^I_j \boldsymbol{\epsilon}_j \tag{4.5}$$

vanishes for $j \neq I$. Hence S and Y generate the centralizer $\mathrm{H} = \mathrm{cen}(Y)$ of Y. The remaining generators of G are divided into two sets

$$X = \{\boldsymbol{\epsilon}_{\boldsymbol{\alpha}} \mid \boldsymbol{\epsilon}_{\boldsymbol{\alpha}} \notin S, \boldsymbol{\alpha} > 0\}, \qquad \bar{X} = \{\boldsymbol{\epsilon}_{\boldsymbol{\alpha}} \mid \boldsymbol{\epsilon}_{\boldsymbol{\alpha}} \notin S, \boldsymbol{\alpha} < 0\}. \tag{4.6}$$

Here $X = \mathfrak{J}\bar{X}^\dagger\mathfrak{J} = \{X_{\boldsymbol{\alpha}}\}$ contains the creation and $\bar{X} = \mathfrak{J}X^\dagger\mathfrak{J} = \{\bar{X}_{\boldsymbol{\alpha}}\}$ the annihilation operators. When these generators are represented by matrices then $X_{\boldsymbol{\alpha}}$ and $\bar{X}_{\boldsymbol{\alpha}}$ are represented by upper- and lower-triangle matrices with zeros on the diagonals.

This splitting of the algebra can be represented by so-called "painted Dynkin diagrams" [105, 106]. Draw the Dynkin diagram of the group G. Label the dots using the Cartan generators $\boldsymbol{\tau}_i$. Put a cross through the dots that represent the Cartan generators $\boldsymbol{\tau}_I$, which are not in S. The Dynkin diagram for the semi-simple part of H is given by the painted Dynkin diagram with the cross dots removed. The cross dots represent the generators $\mathbf{Y}^I$. Let us give two examples: the painted Dynkin diagrams of the cosets $SU(2)/U(1)$ and $SU(5)/[SU(2) \times U(1) \times SU(3)]$ are

 and , (4.7)

respectively.

4.3 BKMU description of G/H

We discuss the construction of Kähler potentials for coset spaces G/H using the method of BKMU [96, 97]. Matter coupling is discussed next, with special attention to sections of line bundles.

The coset G/H has been described by Callan, Coleman, Wess and Zumino [107, 108] by exponentiating of the Hermitean generators

$$R_{\boldsymbol{\alpha}} = \frac{1}{2}(X_{\boldsymbol{\alpha}} + \mathfrak{J}\bar{X}_{\boldsymbol{\alpha}}\mathfrak{J}) \quad \text{and} \quad I_{\boldsymbol{\alpha}} = \frac{1}{2i}(X_{\boldsymbol{\alpha}} - \mathfrak{J}\bar{X}_{\boldsymbol{\alpha}}\mathfrak{J}) \tag{4.8}$$

which forms an unitary element

$$U(x,y) = \exp\left(ix^{\boldsymbol{\alpha}}R_{\boldsymbol{\alpha}} + iy^{\boldsymbol{\alpha}}I_{\boldsymbol{\alpha}}\right) \in \mathrm{G/H}. \tag{4.9}$$

The coordinates $x^{\boldsymbol{\alpha}}$ and $y^{\boldsymbol{\alpha}}$ carry root indices as the elements $X_{\boldsymbol{\alpha}}$ and $\bar{X}_{\boldsymbol{\alpha}}$ do. Using the decomposition of G described in section 4.2 we write an element h of H as an exponential

$$h = e^{i\beta S}e^{i\gamma Y}, \tag{4.10}$$

where $\beta^a, \gamma_I \in \mathbb{R}$ and summation over indices is understood from here on.

The complex coordinates on G/H are introduced by an isomorphism

$$\mathrm{G/H} \longrightarrow \mathrm{G}^{\mathbb{C}}/\hat{\mathrm{H}} \tag{4.11}$$

between G/H and the complexification of G divided by a suitably chosen subgroup $\hat{\mathrm{H}}$ of $\mathrm{G}^{\mathbb{C}}$. The complexification $\mathrm{G}^{\mathbb{C}}$ of G is obtained by replacing the real parameters describing the elements of G by complex ones. The elements $\hat{h}$ of $\hat{\mathrm{H}}$ are represented by

$$\hat{h} = e^{aX}e^{bS}e^{cY}, \tag{4.12}$$

where $a^{\boldsymbol{\alpha}}, b^a, c_I \in \mathbb{C}$ and therefore an element $\xi(z)$ of $\mathrm{G}^{\mathbb{C}}/\hat{\mathrm{H}}$ can be written as

$$\xi(z) = e^{z\bar{X}}, \tag{4.13}$$

where $z^{\boldsymbol{\alpha}} \in \mathbb{C}$. According to [104], $\hat{\mathrm{H}}$ is chosen such that G/H and $\mathrm{G}^{\mathbb{C}}/\hat{\mathrm{H}}$ are isomorphic. Hence there exists a relationship between $U(x, y)$ and $\xi(z)$. The real coordinates (x, y) can therefore be expressed in $z, \bar{z}$. We assume that this isomorphism has been established and write $U(\bar{z}, z)$ for a representative of G/H hereafter. Using this isomorphism, it follows that functions $A^{\boldsymbol{\alpha}}, B^a, K_I$ exist such that

$$U(\bar{z}, z) = \xi(z)e^{A(\bar{z},z)X}e^{B(\bar{z},z)S}e^{-\frac{1}{2}K(\bar{z},z)Y}. \tag{4.14}$$

The representative $U(\bar{z}, z)$ of the equivalence classes of G/H is chosen here such that $B^a(\bar{z}, z)$ an $K_I(\bar{z}, z)$ are always real functions. (The reason for the normalization of the function $K(\bar{z}, z)$ will become clear below, see eq. (4.22).)

The non-linear transformation properties of the coordinates z and $\bar{z}$ of G/H can be defined using ξ [96, 97] or U [107, 108] by

$$\xi(^g z) = g\xi(z)\hat{h}^{-1}(z; g) \quad \text{and} \quad U(^g \bar{z}, ^g z) = gU(\bar{z}, z)h^{-1}(\bar{z}, z; g) \tag{4.15}$$

for any element g of G. The functions $\hat{h}$ and h are chosen such that $\xi(^g z)$ and $U(^g \bar{z}, ^g z)$ are again of the forms given above in eqs. (4.13) and (4.9). Under the composition of two transformations g' and g we find using (4.15) that the non-linear transformation of z respects this composition $^{g'g}z = ^{g'}(^g z)$ and for $\hat{h}$

$$\hat{h}(z; g'g) = \hat{h}(^g z; g')\hat{h}(z; g) \tag{4.16}$$

and similarly for h [76]. Combining the transformation rules (4.15) with the identification of G/H with $\mathrm{G}^{\mathbb{C}}/\hat{\mathrm{H}}$ according to eq. (4.14) shows that

$$e^{^gAX}e^{^gBS}e^{-\frac{1}{2}{^gKY}} = e^{aX}e^{\tilde{A}X}e^{bS}e^{BS}e^{-i\beta S}e^{(c-\frac{1}{2}K-i\gamma)Y} \tag{4.17}$$

where $e^{\tilde{A}X} = (e^{bS}e^{cY})e^{AX}(e^{bS}e^{cY})^{-1}$ and we have used the short-hand notation $^gK = K(^g\bar{z}, ^g z)$, etc. The real functions β^a and γ_J are determined by demanding that ^gB and ^gK are real functions. In particular, this implies that $\gamma_J = \mathrm{Im}\ c_J$, resulting in the Kähler transformation rule [104] for K_J

$$K_J(^g \bar{z}, ^g z) = K_J(\bar{z}, z) - c_J(z; g) - {c_J}^{\dagger}(\bar{z}; g). \tag{4.18}$$

This transformation rule and the reality of $K_J(\bar{z}, z)$ imply that K_J is a Kähler potential.

We now review how Kähler potentials for G/H coset spaces can be obtained and how they relate to the Kähler potentials K_J introduced above [104]. Let $E_{\mathbf{b}}$ be a complex vector space on which $\mathrm{G}^{\mathbb{C}}$ acts as the irreducible representation

$\rho_{\mathbf{b}}(g) : E_{\mathbf{b}} \longrightarrow E_{\mathbf{b}}$ with highest weight $\mathbf{b}$. The so-called BKMU projector $\eta_{\mathbf{b}}$ has the properties [96]

$$(\eta_{\mathbf{b}})^2 = (\eta_{\mathbf{b}})^\dagger = \eta_{\mathbf{b}}\mathfrak{J}_{\mathbf{b}} = \mathfrak{J}_{\mathbf{b}}\eta_{\mathbf{b}} = \eta_{\mathbf{b}} \quad \text{and} \quad \rho_{\mathbf{b}}(\hat{h})\eta_{\mathbf{b}} = \eta_{\mathbf{b}}\rho_{\mathbf{b}}(\hat{h})\eta_{\mathbf{b}}. \tag{4.19}$$

Here $\mathfrak{J}_{\mathbf{b}} = \rho_{\mathbf{b}}(\mathfrak{J})$ denotes the form $\mathfrak{J}$ takes in representation $\mathbf{b}$. The multiplication properties of $\eta_{\mathbf{b}}$ and $\mathfrak{J}_{\mathbf{b}}$ imply that $\eta_{\mathbf{b}}$ only projects on the positive entries of $\mathfrak{J}$. It is shown in refs. [29, 96, 104] that

$$K_{\eta_{\mathbf{b}}}(\bar{z}, z) \equiv \ln \det{}_{\eta_{\mathbf{b}}} \left[\rho_{\mathbf{b}}(\xi^\dagger(\bar{z}))\mathfrak{J}_{\mathbf{b}}\rho_{\mathbf{b}}(\xi(z))\right] \tag{4.20}$$

transforms as a Kähler potential

$$K_{\eta_{\mathbf{b}}}({}^g\bar{z}, {}^g z) = K_{\eta_{\mathbf{b}}}(\bar{z}, z) - \left\{\ln \det{}_{\eta_{\mathbf{b}}} \rho_{\mathbf{b}}\left(\hat{h}(z; g)\right) + \text{h.c.}\right\}, \tag{4.21}$$

using the properties (4.19) of the projector $\eta_{\mathbf{b}}$. Here $\det_{\eta_{\mathbf{b}}}$ denotes the determinant on the subspace on which $\eta_{\mathbf{b}}$ acts as the identity. (We write $\rho_{\mathbf{b}}(\xi) = \xi$, etc. hereafter, when the context fixes the representation.) Furthermore this Kähler potential $K_{\eta_{\mathbf{b}}}$ can be expressed in terms of the Kähler potentials K_J introduced in the previous paragraph as

$$K_{\eta_{\mathbf{b}}}(\bar{z}, z) = \text{tr}_{\mathbf{b}}\left(\eta_{\mathbf{b}} Y^J\right) K_J(\bar{z}, z). \tag{4.22}$$

We use the subscript $\mathbf{b}$ to remind ourselves that the projection operator and the trace are defined with respect to a representation with highest weight $\mathbf{b}$. Using different BKMU projectors within one representation, explicit expressions for the Kähler potentials K_J can be obtained. The set of Kähler potentials $\{K_J\}$ is complete and independent of the representations and projectors used to obtain them [104].

Given an irreducible representation $\mathbf{b}$, we now give an explicit example of a BKMU projector. Denote by $\{|\mathbf{b}, \mathbf{w}\rangle\}$ an orthonormal basis for the complex vector space $E_{\mathbf{b}}$ that is enumerated using the weights $\mathbf{w}$. These vectors satisfy the following properties

$$\begin{gathered} \boldsymbol{\tau}_i|\mathbf{b}, \mathbf{w}\rangle = \mathbf{w}(\boldsymbol{\tau}_i)|\mathbf{b}, \mathbf{w}\rangle, \;\; \langle \mathbf{b}, \mathbf{w}|\mathbf{b}, \mathbf{w}'\rangle = \delta_{\mathbf{w},\mathbf{w}'} \quad \text{and} \\ \boldsymbol{\epsilon}_{-i}|\mathbf{b}, \mathbf{w}\rangle = N_{\mathbf{b},\mathbf{w},i}|\mathbf{b}, \mathbf{w} - \boldsymbol{\alpha}_i\rangle \end{gathered} \tag{4.23}$$

whenever $\mathbf{w} - \boldsymbol{\alpha}_i$ is an element of the weight space with highest weight $\mathbf{b}$. The normalization factor $N_{\mathbf{b},\mathbf{w},i}$ is chosen such that the norm of the vector $|\mathbf{b}, \mathbf{w} - \boldsymbol{\alpha}_i\rangle$ is unity. To analyze the weight space of any representation it is convenient to introduce the Dynkin labels $w_i \equiv \frac{2\langle \mathbf{w}, \boldsymbol{\alpha}_i\rangle}{\langle \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i\rangle}$ of a weight $\mathbf{w}$. Standard Lie group theory [41] teaches us that a Dynkin label w_i is integer and whenever w_i is positive $\mathbf{w} - k\boldsymbol{\alpha}_i$ with $1 \leq k \leq w_i$ is also a weight of representation $\mathbf{b}$. Let $\mathbf{y}$ be any linear combination of elements of Y with positive coefficients. The BKMU operator [96]

$$\eta_{\mathbf{b}}^{\mathbf{y}} = \sum_{\mathbf{w}(\mathbf{y})=\mathbf{b}(\mathbf{y})} |\mathbf{b}, \mathbf{w}\rangle\langle\mathbf{b}, \mathbf{w}| \tag{4.24}$$

projects on the subspace $\mathcal{V}_{\mathbf{b}}^{\mathbf{y}}$ of $E_{\mathbf{b}}$ which contains elements of $E_{\mathbf{b}}$ having the same $\mathbf{y}$-charge as the highest weight vector $|\mathbf{b}, \mathbf{b}\rangle$. We now define the projectors P^J, which allow us to identify the fundamental Kähler potentials K_J directly. Let $\mathbf{b}^J$ be a fundamental weight: a weight which has all its Dynkin labels zero except for the Jth one, which is 1: $(\mathbf{b}^J)_j = \delta_j^J$. Expressed in terms of the Cartan generators we obtain $\mathbf{b}^J = (G^{-1})^{Jj}\boldsymbol{\alpha}_j$. Indeed, we have that $\mathbf{b}^J(\boldsymbol{\alpha}_i) = \delta_i^J$. On the representation space associated with this highest weight $\mathbf{b}^J$ we define the projector P^J by

$$P^J \equiv \eta_{\mathbf{b}^J}^{\mathbf{Y}^J} \tag{4.25}$$

and denote by $\mathcal{V}^J \equiv P^J E_{\mathbf{b}^J}$ the subspace of $E_{\mathbf{b}^J}$ on which P^J projects. As the only non-zero Dynkin label of highest weight $\mathbf{b}^J$ is the Jth one, the weight vector following the highest weight vector is $|\mathbf{b}^J, \mathbf{b}^J - \boldsymbol{\alpha}_J\rangle$. It is easy to see that this vector does not have the same Y^J-charge as $\mathbf{b}^J$, hence it is not contained in $\mathcal{V}^J$, because $\boldsymbol{\alpha}_J(\mathbf{Y}^J) = 1$. This shows that $\mathcal{V}^J$ is one-dimensional and that $P^J = |\mathbf{b}^J, \mathbf{b}^J\rangle\langle\mathbf{b}^J, \mathbf{b}^J|$. Define the Kähler potentials

$$K^J(\bar{z}, z) \equiv K_{P^J} = \ln \det{}_{P^J} \left[\xi^\dagger(\bar{z})\xi(z)\right]. \tag{4.26}$$

We want to use eq. (4.22) to express K^J in the Kähler potentials K_I that form a complete set, therefore we calculate $\operatorname{tr}\left(P^J\mathbf{Y}^I\right)$. As $\mathcal{V}^J$ is one-dimensional this reduces to

$$\operatorname{tr}\left(P^J\mathbf{Y}^I\right) = \mathbf{b}^J(\mathbf{Y}^I) = (G^{-1})^{Jj}\boldsymbol{\alpha}_j(\boldsymbol{\tau}_i)(G^{-1})^{iI} = (G^{-1})^{JI}, \tag{4.27}$$

using the properties (4.23) of the representation vector $|\mathbf{b}^J, \mathbf{b}^J\rangle$. From this it follows that

$$K^J = (G^{-1})^{JI}K_I, \tag{4.28}$$

which gives an explicit expression for K_I. Notice that the Kähler potential $K_{\eta_{\mathbf{b}}}$, hence also the Kähler potentials K^J, is a sum of logarithms of ordinary scalar functions of z and $\bar{z}$ because the subspace $\mathcal{V}^J$ was one-dimensional.

4.3.1 BKMU Matter Coupling

The BKMU construction is not only very useful if one wants to obtain an expression for a Kähler potential of a coset space, but it can also be used to obtain matter couplings to the coset. We have seen that we can obtain matter representations geometrically by introducing them as tangent fields of the manifold, in section 3.3. For a supersymmetric model based on a Kählerian coset space G/H, there is an alternative approach: a supersymmetric non-linear σ-model based on such a coset is equivalent to a linear supersymmetric σ-model with $\mathrm{G}_{\mathrm{local}} \times \mathrm{H}_{\mathrm{global}}$ symmetry [91, 109], where $\mathrm{G}_{\mathrm{local}}$ is a local G symmetry and $\mathrm{H}_{\mathrm{global}}$ a global symmetry. Matter can then be introduced in linear representations of $\mathrm{H}_{\mathrm{global}}$ [29, 74]. The method we describe here is related to those ideas and follows the method described in ref. [98].

For this purpose we consider the BKMU method using any representation **b** of G. (It is often sufficient to take only the fundamental representation to obtain all possible matter couplings.) Let $\eta_A = \mathrm{diag}(\cdots, 0, \mathbb{1}_A, 0, \cdots)$ denote diagonal projection operators on the irreducible H-representations of the representation with the highest weight **b** and satisfying $\sum \eta_A = \mathbb{1}$. According to the remark under eq. (4.6) $\hat{h}(z; g)$ is an upper-triangle matrix,

$$\hat{h} = \begin{pmatrix} \hat{h}_1 & \hat{h}_{12} & \cdots & \cdots \\ 0 & \hat{h}_2 & \hat{h}_{23} & \ddots \\ \vdots & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{4.29}$$

It follows, that for any A we can define [98]

$$\hat{h}_A(z; g) \equiv \eta_A \hat{h}(z; g) \eta_A \tag{4.30}$$

satisfies the same composition property as $\hat{h}$ in eq. (4.16)

$$\hat{h}_A(z; g'g) = \hat{h}_A({}^g z; g') \hat{h}_A(z; g). \tag{4.31}$$

By left-multiplication we obtain the matter representation L_A. It transforms under the action of an isometry $g \in \mathrm{G}$

$${}^g L_A = \hat{h}_A(z; g) L_A. \tag{4.32}$$

To show that these three transformations do indeed define consistent bundles we proceed as follows. Since G/H is a homogeneous space we can reach any point on it by a transformation using a group element $g \in \mathrm{G}$. Therefore we can describe all coordinate transformations as actions of G. The consistency conditions for the bundle of which L_A is a section become

$$\begin{gathered} \hat{h}_A(z; e) = \mathbb{1}_A, \quad \hat{h}_A({}^g z; g^{-1}) = \hat{h}_A(z; g)^{-1}, \\ \hat{h}_A({}^{g_2 g_1} z; g_3) \hat{h}_A({}^{g_1} z; g_2) \hat{h}_A(z; g_1) = \mathbb{1}_A, \end{gathered} \tag{4.33}$$

for $g_1 g_2 g_3 = e$, where e is the identity in the group G. The composition property (4.16) of two group elements shows that these conditions are satisfied. In ref. [76] this is observed for a singlet representation.

For later applications it is important to have a general method to obtain the charges of the matter field L_A. Let u be a $U(1)$ element generated by Y. In any given representation $\xi(z)$ has the identities $\mathbb{1}_A$ on its diagonal, hence we have that $u_A = \hat{h}_A(z;u)$ using (4.15). This implies that the charges of the matter representations obtained in this way are given by standard group theory.

We obtain for L_1 an invariant Kähler potential $K_1 = \bar{L}_1 \chi_1 L_1$. from the metric

$$\chi_1^{-1} \equiv \eta_1 \left(\xi^\dagger(\bar{z})\mathfrak{J}\xi(z)\right)^{-1} \eta_1. \tag{4.34}$$

For L_2 the situation is not so easy as $\hat{h}$ is not a diagonal but only an upper-triangle matrix and therefore χ_2^{-1} does not transform into itself. Using the notation

$$\begin{pmatrix} \chi_1^{-1} & \chi_{12}^{\dashv} \\ \chi_{21}^{\dashv} & \chi_2^{-1} \end{pmatrix} \equiv (\eta_1 + \eta_2) \left(\xi^\dagger(\bar{z})\mathfrak{J}\xi(z)\right)^{-1} (\eta_1 + \eta_2), \tag{4.35}$$

we define a modified χ_2 metric by

$$\tilde{\chi}_2^{-1} \equiv \chi_2^{-1} - \chi_{21}^{\dashv} \chi_1 \chi_{12}^{\dashv}, \tag{4.36}$$

that transforms as $\tilde{\chi}_2^{-1}({}^g\bar{z}, {}^g z) = \hat{h}_2(z;g)\tilde{\chi}_2^{-1}(\bar{z}, z)\hat{h}_2^\dagger(\bar{z};g)$. Therefore $K_2 = \bar{L}_2 \tilde{\chi}_2 L_2$ is an invariant Kähler potential. In a similar fashion metrics $\tilde{\chi}_3, \ldots$ and Kähler potentials $K_3, \ldots$ for $L_3, \ldots$ can be obtained.

In this way all non-linear matter realizations of the isometry group can be obtained. In our general discussion in section 3.3 on matter coupling to supersymmetric σ-models we used tangent vectors of the manifold. They are special tensor products of the matter representations described above. We make this more explicit when we discuss matter coupling in section 4.4.

4.3.2 BKMU Line Bundles

As became clear in section 3.3, non-trivial singlets or sections of line bundles play a central role, hence we discuss sections of line bundles over G/H in detail. We consider the representation with highest weight $\mathbf{b}^J$ again. We discuss two equivalent ways to obtain a section s^J of a complex line bundle: using the covariance of the Kähler potential K^J (4.21) or the method of matter coupling explained in the previous subsection applied to the representation with highest weight $\mathbf{b}^J$. The transformation rule for s^J under the isometries of $\mathcal{M}$ is given by

$$^g s^J = e^{-c^J(z;g)} s^J = \hat{h}^J(z;g) s^J, \tag{4.37}$$

where $\hat{h}^J(z;g) \equiv e^{-c^J(z;g)} = P^J \hat{h}(z;g) P^J$ and $c^J(z;g) = (G^{-1})^{JI} c_I(z;g)$ with c_I used in eq. (4.18). The transformation properties of $\log s^J$ are similar to those the so-called "novino" field [33, 110, 111].

We next determine the charge of a matter representation that transforms like s^J. For this we use the result described in the previous subsection: for a $U(1)$ element u generated by Y we have that

$$P^J u P^J = \hat{h}^J(z;u). \tag{4.38}$$

Therefore we have to compute the $\mathbf{Y}^I$ charge of the highest weight $\mathbf{b}^J$: $\mathbf{b}^J(\mathbf{Y}^I)$ to obtain the charge of s^J. In fact we have already performed this calculation in eq. (4.27). If we compare it to the charge of the coordinate $z^{\boldsymbol{\alpha}_J}$ computed in (4.5) we obtain

$z^{\boldsymbol{\alpha}_J}$	$\boldsymbol{\alpha}_J(\mathbf{Y}^I) = \delta_J^I$
s^J	$\mathbf{b}^J(\mathbf{Y}^I) = (G^{-1})^{JI} = \frac{2}{\langle \boldsymbol{\alpha}_I, \boldsymbol{\alpha}_I \rangle} A^{JI}$

Here $A^{JI} = \frac{\langle \boldsymbol{\alpha}_I, \boldsymbol{\alpha}_I \rangle}{2}(G^{-1})^{JI}$ is the modified inverse Cartan matrix, which can be found in ref. [44] for example. For groups with a simply laced root lattice, we have $A^{JI} = (G^{-1})^{JI}$. The invariant Kähler potential

$$K^J_{\text{line}}(\bar{z}, \bar{s}^J; z, s^J) = \bar{s}^J e^{-K^J(\bar{z},z)} s^J \tag{4.39}$$

can be used for supersymmetric model building purposes [69].

We now show [70], that s^J is a section of a minimal line bundle over G/H in the following two propositions:

A. A set of generating 2-cycles $C_I : \mathbb{C}P^1 \longrightarrow \mathrm{G}^{\mathbb{C}}/\hat{\mathrm{H}}$ of the homology group $H_2(\mathrm{G/H})$ are given by the continuous mappings

$$v \mapsto (0, \ldots, 0, z^{\boldsymbol{\alpha}_I} = v, 0, \ldots, 0). \tag{4.40}$$

B. The 2-cycles C_I and the Kähler potentials K^J satisfy the minimal cocycle conditions

$$\int_{C_I} \omega(K^J) = 2\pi \delta_I^J. \tag{4.41}$$

Hence $\{\omega(K^J)\}$ are the generating Kähler forms of the cohomology group $H^2(\mathrm{G/H}, \mathbb{Z})$.

We now prove these results:
A. The set of 2-cycles $\{C_I\}$ are clearly independent. The map (4.40) is the identity on the restriction of G/H to the submanifold on which it is onto. Therefore the image of the map C_I is compact and it winds only once around the submanifold of G/H. Because G is a simple compact Lie group, the homology group $H_2(\mathrm{G/H})$ is equal to $H_2(\mathrm{G/H}) = H_1(\mathrm{H}) = \mathbb{Z}^n$, where n counts the number of $U(1)$ factors

in H. To show this one uses that $0 \to \mathrm{H} \to \mathrm{G} \to \mathrm{G/H} \to 0$ is a short exact sequence [38], the Künneth formula [37], see eq. (B.15) in appendix B, and the fact that the first homology group of simple compact Lie groups is trivial. As there are as many C_I as $U(1)$ factors in H, it follows that the 2-cycles C_I generate all 2-cycles.
B. We have to calculate $\int_{C_I} \omega(K^J|_{C_I})$ with $K^J = \ln \det_{P^J}(P^J \xi^\dagger \xi P^J)$. Here $K_J|_{C_I}$ denotes the restriction of K^J to the image of C_I. Using (4.40) we find in particular $\xi|_{C_I} = \exp(v\boldsymbol{\epsilon}_{-I})$. The projector P^J in eq. (4.25) satisfies the following properties

$$\boldsymbol{\epsilon}_I \boldsymbol{\epsilon}_{-I} P^J = \delta_I^J P^J \quad \text{and} \quad \Big(\boldsymbol{\epsilon}_{-I}\Big)^n P^J = 0, \quad n > 1.$$

This follows from eq. (4.27) and because only root $\boldsymbol{\alpha}_J$ can be subtracted once from $\mathbf{b}^J$, as the Dynkin labels of $\mathbf{b}^J$ are $(b^J)_j = \delta_j^J$. With these relations, we obtain the following simplification

$$P^J \xi^\dagger(\bar{z}) \xi(z) P^J\big|_{C_I} = \left(1 + \delta_I^J \bar{v} v\right) P^J.$$

The Kähler potential K^J restricted to C_I becomes $K^J|_{C_I} = \delta_I^J \ln\left[1 + \bar{v}v\right]$. By noticing that this is the standard $\mathbb{C}P^1$ Kähler potential, the integral over C_I reduces to the integral over the Kähler form of $\mathbb{C}P^1$. This integral is equal to 2π, hence we obtain (4.41).

4.4 Non-Linear Realization of $SL(N+M,\mathbb{C})$

The previous subsection discussed a very general method to obtain Kähler potentials for a coset with matter coupled to it. In this section we discuss a method that is to a large extent an example of the BKMU method of obtaining non-linear transformations and Kähler potentials. But our starting point is different in this section: we start with a non-linear realization of $SL(M+N;\mathbb{C})$. We find that a real covariant Kähler potential only exists, if we restrict the elements of $SL(M+N;\mathbb{C})$ to $SU_\eta(M,N)$ or one of its subgroups: $SO(2N)$ and $USp(2N)$. The basis of our construction is a transformation rule for a complex $M \times N$-matrix z under the action of an arbitrary element of the special linear group $SL(M+N;\mathbb{C})$. The method explains why all the transformation rules for the different coset spaces based on classical groups are very similar. If we have chosen a given coset, i.e. we have chosen the group of isometries, we can still use the $SL(M+N;\mathbb{C})$ transformations to study the effect of coordinate redefinitions. In particular for a non-compact coset, this allows us to interpolate between two seemingly different representations of the Kähler potential.

Let $g \in SL(M+N;\mathbb{C})$ be an arbitrary element of the special linear group and g^{-1} its inverse, we write

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad g^{-1} = \begin{pmatrix} \alpha^{\dashv} & \beta^{\dashv} \\ \gamma^{\dashv} & \delta^{\dashv} \end{pmatrix}, \tag{4.42}$$

where α, β, γ and δ are $M\times M$-, $M\times N$-, $N\times M$- and $N\times N$-matrices, respectively. The submatrices of the inverse g^{-1} are given by

$$\begin{aligned} \alpha^{\dashv} &= (\alpha - \beta\delta^{-1}\gamma)^{-1}, \qquad \delta^{\dashv} = (\delta - \gamma\alpha^{-1}\beta)^{-1}, \\ \beta^{\dashv} &= -(\alpha - \beta\delta^{-1}\gamma)^{-1}\beta\delta^{-1} = -\alpha^{-1}\beta(\delta - \gamma\alpha^{-1}\beta)^{-1}, \\ \gamma^{\dashv} &= -(\delta - \gamma\alpha^{-1}\beta)^{-1}\gamma\alpha^{-1} = -\delta^{-1}\gamma(\alpha - \beta\delta^{-1}\gamma)^{-1}. \end{aligned} \tag{4.43}$$

To obtain the infinitesimal transformations, one considers infinitesimal deviations from the unit element of $SL(M+N;\mathbb{C})$

$$g = \begin{pmatrix} \mathbb{1}+u & y \\ x & \mathbb{1}-v \end{pmatrix} \quad \text{and} \quad g^{-1} = \begin{pmatrix} \mathbb{1}-u & -y \\ -x & \mathbb{1}+v \end{pmatrix}, \tag{4.44}$$

where u, v, x, y are infinitesimal submatrices. The relative minus in front of the matrices u and v turns out to be useful later. However in the following we are primarily concerned with finite transformations. A non-linear realization is found by defining the matrix $\xi(z)$ as the BKMU parameter by (4.13)

$$\xi(z) = \begin{pmatrix} \mathbb{1} & 0 \\ z & \mathbb{1} \end{pmatrix}. \tag{4.45}$$

Using the same transformation definition as in eq. (4.15), we find that z transforms as

$$\xi(z) \longrightarrow \xi({}^{g}z) = g\xi(z)\hat{h}^{-1}(z;g). \tag{4.46}$$

We obtain that z transforms as

$${}^{g}z = (\gamma + \delta z)\,(\alpha + \beta z)^{-1} \tag{4.47}$$

under the action of g and the matrix $\hat{h}$ takes the form

$$\hat{h}(z;g) = \begin{pmatrix} (\hat{h}_+)^{-1} & \hat{h}_0 \\ 0 & \hat{h}_- \end{pmatrix} = \begin{pmatrix} \alpha + \beta z & \beta \\ 0 & (\delta^{\dashv} - z\beta^{\dashv})^{-1} \end{pmatrix}. \tag{4.48}$$

We have written $(\hat{h}_+)^{-1}$ instead of $\hat{h}_+$ in the matrix $\hat{h}$ for later convenience, at this stage this is merely notation. Notice that it follows from the transformation rule of z that general linear transformations have the same effect as special linear transformations. The subgroup of linear transformations of z is $SL(M,\mathbb{C}) \times SL(N,\mathbb{C})$. Under the composition of two transformations g' and g we find using (4.46) that the non-linear transformation (4.47) respects this composition of transformations ${}^{g'}({}^{g}z) = {}^{g'g}z$ and

$$\hat{h}_-(z;g'g) = \hat{h}_-({}^{g}z;g')\hat{h}_-(z;g) \quad \text{and} \quad \hat{h}_+(z;g'g) = \hat{h}_+(z;g)\hat{h}_+({}^{g}z;g'). \tag{4.49}$$

This is similar to eq. (4.16) which was essential to show that $\hat{h}_A$ can be used to define the transition functions of bundles. In the following we need two projection operators $\eta_\pm$ defined by

$$\eta_+ = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \eta_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{4.50}$$

They are examples of the projectors discussed in subsection 4.3.1. These definitions are consistent with the following notations[1] $(\hat{h}_+)^{-1} = \hat{h}\eta_+ = \eta_+\hat{h}\eta_+$ and $\hat{h}_- = \eta_-\hat{h} = \eta_-\hat{h}\eta_-$. Let $\mathfrak{J} \in SL(M+N;\mathbb{C})$ be a fixed matrix; its additional properties we develop along the way. We define an $M \times M$-matrix function of $z, \bar{z}$ by

$$\tilde{\chi}_{\mathfrak{J}}^{-1}(\bar{z}, z) \equiv \eta_+ \xi^\dagger(\bar{z}) \mathfrak{J} \xi(z) \eta_+ = A + Bz + \bar{z}C + \bar{z}Dz \tag{4.51}$$

and from (4.15) we obtain the transformation property

$$\tilde{\chi}_{\mathfrak{J}}^{-1}(\bar{z}, z) \longrightarrow \tilde{\chi}_{\mathfrak{J}}^{-1}({}^g\bar{z}, {}^g z) = \hat{h}_+^\dagger(\bar{z}; g)\tilde{\chi}_{g^\dagger \mathfrak{J} g}^{-1}(\bar{z}, z)\hat{h}_+(z; g). \tag{4.52}$$

Define the subgroup $SL_{\mathfrak{J}}(M+N;\mathbb{C})$ consisting of elements $g \in SL(M+N;\mathbb{C})$ that leave $\mathfrak{J}$ invariant

$$g^\dagger \mathfrak{J} g = \mathfrak{J} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \mathfrak{J}^{-1} \equiv \begin{pmatrix} A^{\dashv} & B^{\dashv} \\ C^{\dashv} & D^{\dashv} \end{pmatrix}. \tag{4.53}$$

Hence if $g \in SL_{\mathfrak{J}}(M+N;\mathbb{C})$, the function

$$K_{\mathfrak{J}}(\bar{z}, z) = \ln\det \tilde{\chi}_{\mathfrak{J}}^{-1}(\bar{z}, z) \tag{4.54}$$

transforms as a Kähler potential

$$K_{\mathfrak{J}}(\bar{z}, z) \longrightarrow K_{\mathfrak{J}}({}^g\bar{z}, {}^g z) = K_{\mathfrak{J}}(\bar{z}, z) + F(z; g) + \bar{F}(\bar{z}; g), \tag{4.55}$$

with

$$F(z; g) = \ln\det \hat{h}_+(z; g), \qquad \bar{F}(\bar{z}; g) = \ln\det \hat{h}_+^\dagger(\bar{z}; g^\dagger). \tag{4.56}$$

If we want to interpret $K_{\mathfrak{J}}$ as a Kähler potential, $K_{\mathfrak{J}}$ has to be a real function $K_{\mathfrak{J}}(\bar{z}, z) = (K_{\mathfrak{J}}(\bar{z}, z))^\dagger$. This only happens iff $\mathfrak{J}$ is Hermitean $\mathfrak{J}^\dagger = \mathfrak{J}$. The composition rule for F follows directly from that of $\hat{h}_+(z; g)$ given in eq. (4.49)

$$F(z; g'g) = F(z; g) + F({}^g z; g'). \tag{4.57}$$

We define a "finite" real Killing potential $\mathcal{M}(\bar{z}, z; g')$ by

$$2i\,\mathcal{M}(\bar{z}, z; g') = K(\bar{z}, {}^{g'}z) - K({}^{g'}\bar{z}, z) + F(z; g') - \bar{F}(\bar{z}; g'), \tag{4.58}$$

[1]This is slight abuse of notation as $\hat{h}_\pm$ is really a submatrix of $\eta_\pm\hat{h}\eta_\pm$.

parameterized by an arbitrary element $g \in SL(M+N;\mathbb{C})$. Using the transformation property of the Kähler potential (4.55) together with the composition property (4.57) of F, it follows that $\mathcal{M}(\bar{z}, z; g')$ transforms in the adjoint representation $\mathcal{M}({}^g\bar{z}, {}^g z; g') = \mathcal{M}(\bar{z}, z; g^{-1}g'g)$. The usual Killing potentials contracted with the infinitesimal parameters are obtained by inserting an infinitesimal group element (4.44).

The metric associated with $K_{\mathfrak{J}}$ can be written as

$$G_{\underline{\alpha}\alpha} d\bar{z}^{\underline{\alpha}} dz^{\alpha} = \mathrm{tr}\left[\tilde{\chi}_{\mathfrak{J}} d\bar{z} \chi_{\mathfrak{J}} dz\right], \tag{4.59}$$

where we define the $N \times N$-matrix-valued function $\chi_{\mathfrak{J}}$, in analogy to $\tilde{\chi}_{\mathfrak{J}}$ in (4.51), by

$$\chi_{\mathfrak{J}}^{-1}(\bar{z}, z) \equiv \eta_{-}\left(\xi^{\dagger}(\bar{z})\mathfrak{J}\xi(z)\right)^{-1}\eta_{-} = D^{\dashv} - C^{\dashv}\bar{z} - zB^{\dashv} + zA^{\dashv}\bar{z}. \tag{4.60}$$

This can be shown either by direct calculation of the metric in the standard way as the second mixed derivative, or by first proving this for a block-diagonal $\mathfrak{J}$ and showing that the diagonalization procedure has no effect on the metric. This is easy, since under the action of $g \in SL_{\mathfrak{J}}(M, N)$ the differential dz transforms as

$$dz \longrightarrow {}^g(dz) = \hat{h}_{-}(z; g) dz \hat{h}_{+}(z; g) = (\delta^{\dashv} - z\beta^{\dashv})^{-1} dz (\alpha + \beta z)^{-1}, \tag{4.61}$$

and the inverses of the submetrics $\tilde{\chi}_{\mathfrak{J}}, \chi_{\mathfrak{J}}$ transform as eq. (4.52) and as

$$\chi_{\mathfrak{J}}^{-1}(\bar{z}, z) \longrightarrow \chi_{\mathfrak{J}}^{-1}({}^g\bar{z}, {}^g z) = \hat{h}_{-}(z; g)\chi_{\mathfrak{J}}^{-1}(\bar{z}, z)\hat{h}_{-}^{\dagger}(\bar{z}; g). \tag{4.62}$$

It follows that the metric (4.59) is invariant.

Until this point the matrix $\mathfrak{J}$ used in the definitions (4.51) and (4.60) of $\tilde{\chi}_{\mathfrak{J}}$ and $\chi_{\mathfrak{J}}$ can be any Hermitean matrix of $SL(M+N;\mathbb{C})$. However if we want to use eq. (4.54) as a Kähler potential for supersymmetric model building, the resulting kinetic terms have to be positive definite. By Taylor expanding the Kähler potential to its quadratic terms we obtain $\mathrm{tr}(A^{-1}\bar{z}Dz)$. Using a unitary transformation, we can diagonalize $\mathfrak{J}$ with real eigenvalues λ_i. Hence we infer that the quadratic terms are sign definite if $\lambda_i^{-1}\lambda_{i+N}$ all have the same sign. Of course an overall sign can be compensated for. If the diagonalization is followed by an appropriate scale transformation of the coordinates and possibly some relabeling, we bring the matrix $\mathfrak{J}$ into the canonical form

$$\mathfrak{J} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \eta\mathbb{1} \end{pmatrix}, \qquad \eta = \pm 1. \tag{4.63}$$

This shows that we can to restrict ourselves to $SL_{\mathfrak{J}}(M+N;\mathbb{C})$ to $SU_{\eta}(M, N)$ when we want to study the isometries of the metrics $\tilde{\chi}_{\mathfrak{J}}$, $\chi_{\mathfrak{J}}$ or the Kähler potential $K_{\mathfrak{J}}$. Here $\eta = 1$ corresponds to the compact $SU(M+N)$ and $\eta = -1$ to the non-compact $SU(M, N)$ special unitary group. From now on we assume that we have chosen this canonical form of $\mathfrak{J}$ and consider $SU_\eta(M,N)$ or one of its subgroups only. By putting further restrictions on the group elements g we can reduce the isometry group to subgroups, like $SO(2N)$ and $USp(2N)$. But the form of the metrics $\tilde{\chi}_\mathfrak{J}$, $\chi_\mathfrak{J}$ and Kähler potential do not change under this; they always take the form

$$\begin{gathered}\tilde{\chi}_\eta(\bar{z},z) = (1\!\!1 + \eta\bar{z}z)^{-1}\,, \quad \chi_\eta(\bar{z},z) = (1\!\!1 + \eta z\bar{z})^{-1}\,, \\ K_\eta(\bar{z},z) = \eta \ln\det\tilde{\chi}_\eta^{-1} = \eta\ln\det\chi_\eta^{-1}\end{gathered} \tag{4.64}$$

in the canonical basis. However in the more specific cases there are restrictions on the coordinates z as we see later: that is the coordinates z parameterize a submanifold of $SU_\eta(M,N)/S[U(M)\times U(N)]$.

Even though the $SL(M+N;\mathbb{C})$ group is not the isometry group, it is still worthwhile to know its action on the fields, as it can be used to describe field redefinitions. We give an example of this now. Above we used that the fact, that we can take B and C in the matrix $\mathfrak{J}$ to zero by a unitary transformation. Sometimes we can also do the opposite: set A and D to zero. To analyze the situation we start with $\mathfrak{J}$ in the canonical form and perform an arbitrary transformation g of $SL(M+N;\mathbb{C})$ on it

$$g^\dagger \begin{pmatrix} 1\!\!1 & 0 \\ 0 & \eta 1\!\!1 \end{pmatrix} g = \begin{pmatrix} \bar{\alpha}\alpha + \eta\,\bar{\gamma}\gamma & \bar{\alpha}\beta + \eta\,\bar{\gamma}\delta \\ \bar{\beta}\alpha + \eta\,\bar{\delta}\gamma & \bar{\beta}\beta + \eta\,\bar{\delta}\delta \end{pmatrix}. \tag{4.65}$$

Hence to remove the A and D entries of this matrix, we require that

$$\bar{\alpha}\alpha + \eta\,\bar{\gamma}\gamma = 0 \quad \text{and} \quad \bar{\beta}\beta + \eta\,\bar{\delta}\delta = 0. \tag{4.66}$$

There is no solution $g \in SL(M+N;\mathbb{C})$ of these equations when $\eta = 1$. On the other hand in the case $\eta = -1$ and $M = N$ we can use

$$g = \frac{i}{\sqrt{2}} \begin{pmatrix} -1\!\!1 & -1\!\!1 \\ -1\!\!1 & 1\!\!1 \end{pmatrix} \quad \Longrightarrow \quad \mathfrak{J} = \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix} \tag{4.67}$$

such that the Kähler potential takes the form

$$K_{no-sc} = \ln\det(z+\bar{z}). \tag{4.68}$$

This Kähler potential is similar to no-scale type [112]. The low energy effective actions for the moduli sectors of string theory often take this form [113, 114].

4.4.1 Classic Cosets

Until this point our discussion was general, in the sense that we only demanded that we construct isometries of the metrics $\tilde{\chi}_\mathfrak{J}$ and $\chi_\mathfrak{J}$ without any reference to a particular coset space. We saw that we only obtain isometries of these metrics if we restrict the transformations to be unitary: $g \in SU_\eta(M,N)$. It is now easy to describe non-linear realizations of (classic) groups, that are subgroups of $SU_\eta(M,N)$. For this we only have to describe how these subgroups are embedded in the unitary group $SU_\eta(M,N)$. We have summarized our results in table 4.1. We describe the entries of this table which are partly taken from ref. [43]. A complete classification of Kähler cosets can be found in ref. [115]. A discussion on $SO(2N)/U(N)$, $Sp(2N)/U(N)$ cosets can also be found in refs. [30, 116, 117]. Non-compact coset $Sp(2N)/U(N)$ and $SU(M,N)/[SU(M) \times U(1) \times SU(N)]$ as generalizations of no-scale models are discussed in ref. [118]. In chapter 6 we discuss various aspects of the coset $SO(2N)/U(N)$.

The classic groups are real or complex groups that satisfy certain Hermitean conjugation and transposition properties

$$g^\dagger \mathfrak{J} g = \mathfrak{J} \quad \text{and} \quad g^T \mathfrak{K}\, g = \mathfrak{K} \tag{4.69}$$

where $\mathfrak{J}$ and $\mathfrak{K}$ are fixed matrices. We discriminate between the special unitary (SU), special orthogonal (SO), symplectic (Sp) and unitary symplectic (USp) groups. Furthermore, with the parameter $\eta = \pm 1$ we make a distinction between compact ($\eta = 1$) and non-compact ($\eta = -1$) groups. We require that a compact $SO(2) \cong U(1)$-factor is contained in a maximal subgroup H of these groups. For this reason we do not consider the non-compact special orthogonal group $SO(N,N)$ here, as the non-compact $SO(1,1)$ corresponds to Lorentz transformations that are not bounded. The compact $U(1)$ factor is needed to ensure that the resulting coset space is Kähler. Because of its importance we give the $U(1)$ embedding explicitly. For the real groups $SO(2N)$ and $Sp(2N)$ the $U(1)$ is not realized in a diagonal way. By making a similarity transformation [41, 43]

$$g_D = V g V^\dagger, \quad g = V^\dagger g_D V \quad \text{and} \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & i\mathbb{1} \\ i\mathbb{1} & \mathbb{1} \end{pmatrix}, \tag{4.70}$$

using the unitary matrix V, the $U(1)$ is turned into a diagonal form. Here the subscript D is used to indicate that g_D, for example, is considered in the basis where the $U(1)$ is diagonal. For real groups, that are embedded in $SL(M+N;\mathbb{R})$, we have $g^\dagger = g^T$ which in the diagonal $U(1)$ basis becomes

$$g_D^\dagger = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} g_D^T \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{4.71}$$

The transposition properties change as

$$g_D^T \mathfrak{K}_D g_D = \mathfrak{K}_D = \left(V^\dagger\right)^T \mathfrak{K}\, V^\dagger. \tag{4.72}$$

For this it is crucial that we have embedded the real groups $Sp(2N)$ and $SO(2N)$ in the special unitary group $SU(N,N)$ and $SU(2N)$ respectively; otherwise the multiplication with i has no meaning. From now on we work in the basis where the

$U(1)$ factor is diagonal. We can represent any element of any of these groups as a unitary matrix $g_D = e^{a_D}$, that is obtained by exponentiating an anti-Hermitean algebra element a_D. The group definition properties (4.69) can be written down for the algebra elements a_D as well

$$a_D^\dagger = -\mathfrak{J} a_D \mathfrak{J}^{-1} \quad \text{and} \quad a_D^T = -\mathfrak{K}_D a_D \mathfrak{K}_D^{-1}. \tag{4.73}$$

Using these properties it is possible to give a unique representation of the algebra elements a_D. For the different groups we give this representation in the ninth and tenth rows of table 4.1.

Notice that algebra elements of $Sp(2N)$ and $USp(N,N)$ have the same representation in the basis where the $U(1)$ is diagonal; therefore their corresponding cosets are isomorphic. From this representation of the algebra, it is easy to see what the symmetry properties are of the coordinates z for the different cosets. For the non-compact coset, the coordinates in addition satisfy $\mathrm{tr}(z\bar{z}) < 1$, so that the Kähler potential and metrics in (4.64) do not diverge. Notice that for the USp, Sp and SO cosets the submetrics $\tilde{\chi}_\eta, \chi_\eta$ are each others transposed $\tilde{\chi}_\eta = \chi_\eta^T$, because the coordinates z are either symmetric or anti-symmetric.

4.4.2 $SL(M+N;\mathbb{C})$ Matter Representations

Matter coupling is the next topic to discuss. As we want to interpret $SU_\eta(M,N)$ or one of its subgroups as the internal symmetry group of the models we construct, this implies that all matter fields should behave as well-defined representations of $SU_\eta(M,N)$. However as far as the transformation properties go, they actually apply to any $SL(M+N;\mathbb{C})$ transformation. We first use the general tangent bundle approach and next apply the method explained in subsection 4.3.1 with the vector representation of $SU_\eta(M,N)$. We finish this subsection with a discussion of the line bundle.

To obtain a section of the tangent bundle we define the transformation of the tangent space vector T, in analogy to (4.61), by

$$^{g}T = \hat{h}_-(z;g) T \hat{h}_+(z;g) = (\delta^{\dashv} - z\beta^{\dashv})^{-1} T (\alpha + \beta z)^{-1}. \tag{4.74}$$

A section C of the cotangent bundle transforms as

$$^{g}C = \left(\hat{h}_+(z;g)\right)^{-1} C \left(\hat{h}_-(z;g)\right)^{-1} = (\alpha + \beta z) C (\delta^{\dashv} - z\beta^{\dashv}). \tag{4.75}$$

If we take $g \in SU_\eta(M,N)$, we obtain the following invariants for the sections of the tangent and cotangent bundles

$$\mathrm{tr}\left[\tilde{\chi}_{\mathfrak{J}} \bar{T} \chi_{\mathfrak{J}} T\right] \quad \text{and} \quad \mathrm{tr}\left[(\chi_{\mathfrak{J}})^{-1} \bar{C} (\tilde{\chi}_{\mathfrak{J}})^{-1} C\right]. \tag{4.76}$$

Group G	$SU_\eta(M,N)$	$USp_\eta(N,N)$	$Sp(2N)$	$SO(2N)$
$\eta =$	± 1	± 1	-1	1
Compact subgroup H	$S[U(M)\times U(N)]$	$U(N)$	$U(N)$	$U(N)$
$g \in$	$SL(M+N;\mathbb{C})$	$SL(2N;\mathbb{C})$	$SL(2N;\mathbb{R})$	$SL(2N;\mathbb{R})$
$g^\dagger \mathfrak{I} g = \mathfrak{I} =$	$\begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}$	–	–
$g^T \mathfrak{K} g = \mathfrak{K} =$	–	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$U(1)$ embedding	$\begin{pmatrix} e^{i\frac{N\theta}{P}} & 0 \\ 0 & e^{-i\frac{M\theta}{P}} \end{pmatrix}$	$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$	$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
$g_D^T \mathfrak{K}_D g_D = \mathfrak{K}_D =$	–	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$g_D = e^{a_D},\ a_D =$	$\begin{pmatrix} u & -\eta x^\dagger \\ x & -v \end{pmatrix}$	$\begin{pmatrix} u & -\eta x^\dagger \\ x & -u^T \end{pmatrix}$	$\begin{pmatrix} u & x^\dagger \\ x & -u^T \end{pmatrix}$	$\begin{pmatrix} u & -x^\dagger \\ x & -u^T \end{pmatrix}$
Restrictions	$u^\dagger = -u,\ v^\dagger = -v$ $\operatorname{tr} u = \operatorname{tr} v$	$u^\dagger = -u,\ x^T = x$	$u^\dagger = -u,\ x^T = x$	$u^\dagger = -u,\ x^T = -x$
$z \in$ G/H, $z^{ij} \in \mathbb{C}$	–	$z^T = z$	$z^T = z$	$z^T = -z$

Table 4.1: This table gives an overview of the (classical) Lie-groups that can be embedded into $SU_\eta(M,N)$. With the parameter η we distinguish between compact ($\eta = 1$) and non-compact ($\eta = -1$) groups. For these Lie-groups the non-linear $SL(M+N;\mathbb{C})$ transformation rules given in this section can be used directly. $P = \gcd(M,N)$ is defined as the greatest-common-divisor of M and N. If the $U(1)$ is not diagonal, we have to perform a special unitary transformation to make it diagonal; this may change the transposition properties. The Hermitean form of an element of the algebra after possible diagonalization is denoted by a_D. The matrices u, v, x are all taken to be complex, their additional properties are given in the table. The last row gives the restrictions on the coset coordinates.

Next we construct subbundles of the tangent bundle. To do this we notice that the transformation rule (4.61) for the differential dz factorizes. Using this we define the sections L and R by the transformation rules

$$^{g}L = \hat{h}_-(z;g)L = (\delta^{\dashv} - z\beta^{\dashv})^{-1}L, \qquad ^{g}R = R\,\hat{h}_+(z;g) = R\,(\alpha+\beta z)^{-1}. \quad (4.77)$$

The consistency conditions, for the bundles of which L and R are sections, are satisfied as they are an example of $\hat{h}_A$ in eq. (4.33). Using that the metric of the tangent bundle (4.59) factorizes as well, we obtain the following $SU_\eta(M,N)$-invariants

$$\bar{L}\chi_{\mathfrak{Z}}L \quad \text{and} \quad R\tilde{\chi}_{\mathfrak{Z}}\bar{R}. \quad (4.78)$$

We will discuss tensor products of these types of matter representations extensively when we consider matter coupling to $SO(2N)/U(N)$ in chapter 6.

We now turn to the construction of the minimal complex line bundles. A section S of a complex line bundle can be defined to transform as

$$^{g}S = \det\hat{h}_+(z;g)S = \det\hat{h}_-(z;g)S. \quad (4.79)$$

Here we have used that $\det\hat{h}_+ = \det\hat{h}_-$, which follows from (4.46) as the element $g \in SU_\eta(M,N)$ and $\det\xi(z) = 1$. The consistency (4.33) of this complex line bundle follows directly from (4.49) and the properties of the determinant. To determine which power of the minimal line bundle we have obtained, we calculate the integral over the corresponding Kähler form

$$\int_{C_2} \omega(K) = 2\pi\, n, \quad \text{with} \quad n \in \mathbb{Z}, \quad (4.80)$$

over a generating two-cycle C_2. If $n = \pm 1$, K is a Kähler potential corresponding to a minimal line bundle.

At this stage we have to make a distinction between the different cosets $SU(M+N)/S[U(M)\times U(N)]$, $SO(2N)/U(N)$ and $USp(2N)/U(N)$. We first turn to a Grassmannian coset $SU(M+N)/S[U(M)\times U(N)]$. Let v be the complex coordinate of the stereo-graphic projection of the complex projective line $\mathbb{C}P^1$. We define a generating two-cycle by the embedding of $\mathbb{C}P^1$ in the coset by taking all the coordinates z^{ij} zero except for one which is equal to v. Now since the Kähler potential restricted to this embedding to $\mathbb{C}P^1$ is given by $K(\bar{z},z)|_{\mathbb{C}P^1} = \ln(1+\bar{v}v) = K_{\mathbb{C}P^1}(\bar{v},v)$, which is the Kähler potential of $\mathbb{C}P^1$ that satisfies $\int_{\mathbb{C}P^1}\omega(K_{\mathbb{C}P^1}) = 2\pi$, it follows that we have obtained a minimal line bundle.

Next we discuss the compact $USp(2N)/U(N)$ and $SO(2N)/U(N)$ coset spaces. The coordinates of these spaces satisfy $z^T = z$ and $z^T = -z$, respectively, see table 4.1. Therefore for fixed indices i,j we take the symmetrization into account: $z^{ij} = \pm z^{ji} = v$, putting all other z^{kl} to zero. Hence we find in these cases that $K(\bar{z}, z)|_{\mathbb{C}P^1} = 2\ln(1 + \bar{v}v) = 2K_{\mathbb{C}P^1}(\bar{v}, v)$, so that $n = 2$ in eq. (4.80). This implies that the section S is the square of the minimal line bundle. Since the Kähler potential of a coset is unique up to a normalization factor, it follows that a section of a minimal line bundle over $USp(2N)/U(N)$ or $SO(2N)/U(N)$ is given by

$$ {}^{g}S = \Big(\det \hat{h}_+(z; g)\Big)^{\frac{1}{2}} S = \Big(\det \hat{h}_-(z; g)\Big)^{\frac{1}{2}} S. \tag{4.81} $$

The only possible ambiguity for a global definition resides in the square root; it can be removed by using the BKMU construction with the representation with highest weight that has all its Dynkin label zero except for the Nth one, see section 4.3.2.

We now determine the relative charges of the coordinates z, the matter fields L and R, and the sections of the minimal line bundles. Again we first discuss the Grassmannian cosets and after that the cosets $USp(2N)/U(N)$ and $SO(2N)/U(N)$. The $U(1)$ factor

$$ u_\theta = \begin{pmatrix} e^{-iN\theta/P}\mathbb{1} & 0 \\ 0 & e^{iM\theta/P}\mathbb{1} \end{pmatrix} \tag{4.82} $$

commutes with the subgroup $SU(M) \times SU(N)$ within $SU_\eta(M, N)$. Here we defined $P = \gcd(M, N)$ as the greatest-common-divisor of M and N. The smallest period of this $U(1)$ is $\theta = 2\pi$, since the integers N/P and M/P are relatively prime by construction. It follows that the coordinates z have charge $(M+N)/P$ in this normalization. For the matter couplings L and R we find the charges N/P and M/P, respectively. The section of the minimal line bundle has a charge MN/P. For the cosets $USp(2N)/U(N)$ and $SO(2N)/U(N)$ we always obtain integer charges, if we choose a slightly different normalization for u_θ given by

$$ u_\theta = \begin{pmatrix} e^{-i2\theta}\mathbb{1} & 0 \\ 0 & e^{i2\theta}\mathbb{1} \end{pmatrix}. \tag{4.83} $$

In this case L and R have the same charge 2 and the section of the minimal line bundle has charge N, while the charge of the coordinates is 4. The charge of the line bundle is N in stead of $2N$ because of the square root in eq. (4.81).

4.5 Non-Linear Realization of Symmetric Algebras

In section 4.3 we discussed the general construction of non-linear realizations of algebras, Kähler potentials and matter couplings. In the previous section we have discussed a powerful method to obtain Kähler potentials for classic coset spaces and matter coupled to them. We now discuss an alternative way to do this, that can be applied to a coset space with a symmetric algebra: one for which the commutator of two broken creation operators is always zero. A Kähler potential for the $E_6/[SO(10) \times U(1)]$ coset [32] can be obtained using this method, as we will see in chapter 7.

We consider a Kählerian coset G/H where G is a group generated by a symmetric algebra and H, as before, contains the linear symmetries. First we write the algebra together with its Jacobi conditions. Next we obtain a linear realization of the algebra in terms of complex scalars by requiring that these scalars transform as the adjoint representation of G. To find a non-linear realization a "gauge" choice is made for some of the scalars by giving them H-invariant constant values. This choice changes some transformation rules and the algebra of the Killing vectors does not close anymore on the remaining complex scalar fields. To make the algebra close again, some transformation rules have to be altered; they become non-linear. Since the goal is the construct transformation rules of chiral multiplets in supersymmetric models, they should be holomorphic in the fields.

For later convenience, we slightly change our notation for the generators of the algebra in section 4.2. Y^I are the $U(1)$ factor generators, S_a are the remaining generators of H and finally X^α and $\bar{X}^{\underline{\alpha}}$ correspond to the broken symmetries. Here α are not to be confused with the root indices $\boldsymbol{\alpha}$ used before. The generators here are Hermitean and using the Killing metric $\eta_{\underline{\alpha}\alpha}$ for the broken generators we denote $\bar{X}_\alpha = \bar{X}^{\underline{\beta}}\eta_{\underline{\beta}\alpha}$. We only consider symmetric algebras, for which the commutators of all X^α with X^β vanish. The commutation relations for a symmetric algebra can be written in the form

$$
\begin{aligned}
&[Y_I, Y_J] = 0, && [Y_I, S_a] = 0,\\
&[Y_I, X^\alpha] = A_{I\ \beta}^{\ \alpha} X^\beta, && [Y_I, \bar{X}_\alpha] = -\bar{X}_\beta A_{I\ \alpha}^{\ \beta},\\
&[S_a, X^\alpha] = B_{a\ \beta}^{\ \alpha} X^\beta, && [S_a, \bar{X}_\alpha] = -\bar{X}_\beta B_{a\ \alpha}^{\ \beta},\\
&[X^\alpha, X^\beta] = [\bar{X}_\alpha, \bar{X}_\beta] = 0, && [S_a, S_b] = iD_{ab}{}^c S_c,\\
&[X^\alpha, \bar{X}_\beta] = A_{I\ \beta}^{\ \alpha} Y^I + B_{a\ \beta}^{\ \alpha} S_a,
\end{aligned}
\tag{4.84}
$$

where all of the structure coefficients $D_{ab}{}^c$ are chosen real. Summation over repeated indices a, α and I is understood. The structure coefficients $A_{I\ \beta}^{\ \alpha}$ and $B_{a\ \beta}^{\ \alpha}$ satisfy

$$
\eta^{-1} A_I^\dagger \eta = A_I \quad \text{and} \quad \eta^{-1} B_a^\dagger \eta = B_a. \tag{4.85}
$$

The structure coefficients are chosen such that many Jacobi identities are fulfilled automatically, but there are some remaining consistency relations

$$
\begin{gathered}
B_{[a\gamma}^{\ \alpha}B_{b]\beta}^{\ \gamma} = iD_{ab}{}^{c}B_{c\,\beta}^{\ \alpha}, \qquad D_{d(a}{}^{e}D_{cb)_c}{}^{d} = 0, \\
A_{I\,\gamma}^{[\alpha}A_{I\,\delta}^{\ \beta]} + B_{a\,\gamma}^{[\alpha}B_{a\,\delta}^{\ \beta]} = 0.
\end{gathered} \tag{4.86}
$$

The subscript $(\dots)_c$ denotes that we have to sum over all cyclic permutations of the indices enclosed. The last Jacobi identity is crucial for the closure of the algebra realized on the coordinates of the coset space G/H as we see shortly. A couple of additional remarks concerning the structure coefficients are in order. As the structure coefficients A_I and B_a correspond to linear symmetries, they can be taken block-diagonal when interpreted as matrices. Each block corresponds to an H-irreducible sector of the algebra, therefore the blocks of A_I are proportional to the identity of that block, as they correspond to the charges Y_I. The basis $\{S_a\}$ can always be chosen such that the Killing metric η_{ab} is proportional to the identity δ_{ab}. In this basis the structure coefficients $D_{ab}{}^{c}$ are completely anti-symmetric. In the following we assume that we have made these choices.

The tensor

$$
M_{\gamma\delta}^{\alpha\beta} = A_{I\,\gamma}^{\ \alpha}A_{I\,\delta}^{\ \beta} + B_{a\,\gamma}^{\ \alpha}B_{a\,\delta}^{\ \beta} \tag{4.87}
$$

has various properties that are crucial for the construction of the non-linear realization of the algebra. From the last Jacobi identity in eq. (4.86), it follows that the tensor $M_{\gamma\delta}^{\alpha\beta}$ is symmetric under the interchange of (α,β) and (γ,δ) separately. The tensor M satisfies

$$
M_{\gamma\,\beta}^{\kappa\eta}M_{\kappa\,\alpha}^{\sigma\,\delta}\,x^{\gamma}x^{\beta}x^{\alpha} = M_{\gamma\,\beta}^{\kappa\sigma}M_{\kappa\alpha}^{\delta\eta}\,x^{\gamma}x^{\beta}x^{\alpha}, \tag{4.88}
$$

where x^{α} is an arbitrary vector of complex numbers. Together with the symmetry property of the tensor M under interchange of indices, this implies that $M_{\delta\epsilon}^{\alpha\beta}M_{\phi\rho}^{\gamma\delta}$ is completely symmetric in α,β,γ, if it is symmetrized over ϵ,ϕ,ρ. To prove eq. (4.88), we define the tensor N by

$$
N_{\gamma\,\beta\alpha}^{\sigma\eta\,\delta} \equiv M_{\gamma\,\beta}^{\kappa\eta}M_{\kappa\,\alpha}^{\sigma\,\delta} - M_{\gamma\,\alpha}^{\kappa\,\delta}M_{\kappa\,\beta}^{\sigma\eta}, \tag{4.89}
$$

following ref. [32]. Using the symmetry property of the tensor $M_{\gamma\,\beta}^{\kappa\eta}$ under the interchange of indices, one can show that

$$
N_{\gamma\,\beta\alpha}^{\sigma\eta\,\delta} = N_{\alpha\,\beta\gamma}^{\sigma\eta\,\delta} - N_{\beta\,\alpha\gamma}^{\sigma\eta\,\delta} - N_{\beta\alpha\gamma}^{\delta\sigma\eta} + N_{\beta\alpha\gamma}^{\eta\sigma\delta}. \tag{4.90}
$$

The tensor N can also be expressed as

$$
N_{\gamma\,\beta\alpha}^{\sigma\eta\,\delta} = -B_{a\,\beta}^{\ \eta}B_{b\,\alpha}^{\ \delta}\left(B_{a\,\kappa}^{\ \sigma}B_{b\,\gamma}^{\ \kappa} - B_{b\,\kappa}^{\ \sigma}B_{a\,\gamma}^{\ \kappa}\right) = -iD_{ab}{}^{c}B_{a\,\beta}^{\ \eta}B_{b\,\alpha}^{\ \delta}B_{c\,\gamma}^{\ \sigma}, \tag{4.91}
$$

using one of the Jacobi identities (4.86). As $D_{ab}{}^{c}$ is complete anti-symmetric, it follows that

$$
N_{\alpha\,\beta\gamma}^{\sigma\eta\,\delta} = -N_{\beta\,\alpha\gamma}^{\eta\sigma\,\delta} = -N_{\alpha\,\gamma\beta}^{\sigma\,\delta\,\eta}. \tag{4.92}
$$

Combining these anti-symmetry properties with eq. (4.90) gives

$$N_{\gamma\,\beta\alpha}^{\sigma\eta\,\delta} + N_{\gamma\beta\alpha}^{\delta\sigma\eta} + N_{\gamma\,\beta\alpha}^{\eta\delta\,\sigma} = 0. \tag{4.93}$$

Eq. (4.88) is obtained from this by contracting with $x^\alpha x^\beta x^\gamma$ and using the definition (4.89) of N. This property can be generalized to the statement that the expression

$$M_{\beta_1\delta_1}^{\alpha_1\alpha} M_{\beta_2\delta_2}^{\alpha_2\beta_1} \dots M_{\delta\delta_n}^{\alpha_n\beta_{n-1}}\, x^{\delta_1} \dots x^{\delta_n} x^\delta \tag{4.94}$$

is symmetric in $(\alpha, \alpha_1, \dots, \alpha_n)$.

We introduce complex scalars that transform in the adjoint of G: we assign to the generators Y, S_a, X^α and $\bar{X}_\alpha$ the complex scalar fields y, s_a, z^α and x_α respectively. Their transformation rules are defined by the replacement

$$[A, B] = C \longrightarrow [A, b] = c, \tag{4.95}$$

where A, B, C are algebra elements and a, b, c are complex scalars in the adjoint representation of G. It is easy to see that the linear relation of the algebra closes on y, s_a, z^α and x_α by writing the Jacobi identity as

$$[A, [B, C]] - [B, [A, C]] = [[A, B], C], \tag{4.96}$$

as the only thing we have to do to obtain this linear realization is to replace the generator C by the scalar c.

We now construct a non-linear realization of this algebra. We do this by restricting the scalars y_I, s_a, x_α to constant H-invariant real "vacuum expectation" values:

$$y_I \;=\; y_I^0/f, \qquad s_a \;=\; 0, \qquad x_\alpha = 0, \tag{4.97}$$

where $f = M_\sigma^{-1}$ is a parameter to give y_I a dimension of mass while y_I^0 is dimensionless. We call this procedure gauge-fixing although it is a slightly improper use of this notion. The transformations of z^α under the action of Y_I or S_a remain the same: $[Y_I, z^\alpha] = A_{I\,\beta}^{\;\alpha} z^\beta$ and $[S_a, z^\alpha] = B_{a\,\beta}^{\;\alpha} z^\beta$. The first thing that changes, is that the transformation of z^α under the action of $\bar{X}_\beta$ becomes non-linear

$$[\bar{X}_\beta, z^\alpha] = -\frac{1}{f} A_\beta^\alpha, \tag{4.98}$$

where $A_\beta^\alpha = y_I^0 A_{I\,\beta}^{\;\alpha}$. We assume that the inverse A^{-1} of the matrix A exists. Different choices of the constants y_I^0 can lead to different Kähler potentials corresponding to different complex structures [29]. Because of this transformation property, the algebra

$$[X^\alpha, [\bar{X}_\beta, z^\gamma]] - [\bar{X}_\beta, [X^\alpha, z^\gamma]] = [[X^\alpha, \bar{X}_\beta], z^\gamma] \tag{4.99}$$

does not close anymore unless the commutator $[X^\alpha, z^\gamma]$ is quadratic in z [119]. This is necessary as $\bar{X}_\beta$ removes a z and the right-hand side is linear in z. If we define

$$[X^\alpha, z^\beta] \,=\, \frac{1}{2} f\, M^{\alpha\beta}_{\gamma\delta}\, (A^{-1}z)^\gamma z^\delta, \tag{4.100}$$

the algebra closes. The choice whether to write $(A^{-1}z)^\gamma z^\delta$ or $z^\gamma(A^{-1}z)^\delta$ is irrelevant as $M^{\alpha\beta}_{\gamma\delta}$ is symmetric in γ and δ and A^{-1} is a diagonal matrix, as A_I was chosen to be proportional to the identity on H-irreducible blocks. Using the fact that A^γ_β is constant and the symmetry properties of the tensor M (4.87), the left-hand side of eq. (4.99) becomes

$$-[\bar{X}_\beta, [X^\alpha, z^\gamma]] = M^{\alpha\gamma}_{\beta\delta} z^\delta = \left(A_{I\,\beta}^{\;\alpha} A_{I\,\delta}^{\;\gamma} + B_{m\,\beta}^{\;\alpha} B_{m\,\delta}^{\;\gamma}\right) z^\delta. \tag{4.101}$$

This is the same as the right-hand-side:

$$[[X^\alpha, \bar{X}_\beta], z^\gamma] = \left(A_{I\,\beta}^{\;\alpha} A_{I\,\delta}^{\;\gamma} + B_{m\,\beta}^{\;\alpha} B_{m\,\delta}^{\;\gamma}\right) z^\delta. \tag{4.102}$$

Having determined the non-linear transformations, we now construct a Kähler potential K_σ that is covariant under these transformations as a power series, starting from the lowest order Kähler potential $K^0_\sigma = \bar{z}_\alpha z^\alpha$. We show that the covariant Kähler potential K_σ is given by

$$K_\sigma = \bar{z} K(Q) z \quad \text{with} \quad K(Q) = Q^{-1} \ln(\mathbb{1} + Q), \tag{4.103}$$

where the appropriate contractions of indices are assumed to be understood and the matrix Q is defined by

$$Q^\beta_\alpha \;\equiv\; \frac{1}{2} f^2\, M^{\beta\delta}_{\alpha\gamma}\, (A^{-1}z)^\gamma (\bar{z}A^{-1})_\delta. \tag{4.104}$$

We start with the ansatz that the covariant Kähler potential K_σ can be written in terms of an analytic function $K(Q)$ of the matrix Q. Clearly $K(Q)$ is invariant under all linear symmetries. If the values of $z, \bar{z}$ are small, the Kähler potential K_σ should reduce to K^0_σ, therefore $K(0) = \mathbb{1}$. We notice that

$$\frac{1}{2} f^2 \bar{z} Q^n z = M^{\alpha_1\alpha_0}_{\beta_1\delta_1} M^{\alpha_2\beta_1}_{\beta_2\delta_2} \ldots M^{\alpha_n\beta_{n-1}}_{\delta_0\delta_n}\, T^{\delta_0}_{\alpha_0} T^{\delta_1}_{\alpha_1} \ldots T^{\delta_n}_{\alpha_n} \tag{4.105}$$

where $T^\delta_\alpha = \frac{1}{2} f^2\, \bar{z}_\alpha z^\delta$ transforming under the non-linear symmetries $\bar{\epsilon}_\alpha \delta^\alpha$, denoted by δ^α with parameter $\bar{\epsilon}_\alpha$, as

$$\bar{\epsilon}_\alpha \delta^\alpha \left(T^\gamma_\beta\right) = i\frac{1}{2} f\, \bar{\epsilon}_\alpha \left(A^\alpha_\beta z^\gamma \;+\; \frac{1}{2} f^2\, M^{\alpha\gamma}_{\delta\epsilon} (A^{-1}z)^\delta z^\epsilon \bar{z}_\beta \right). \tag{4.106}$$

Using the symmetry properties (4.94) of M under symmetrizations and that A is proportional to the identity on H-invariant blocks, we find that

$$\bar{\epsilon}_\alpha \delta^\alpha (\bar{z} K(Q) z) = i \frac{1}{f} \bar{\epsilon} A \left((\mathbb{1} + Q) \frac{\mathrm{d}}{\mathrm{d}Q} Q K(Q) \right) z \tag{4.107}$$

for the analytical function $K(Q)$. Now as K_σ has to transform covariantly in a holomorphic function of z, the expression in between brackets in eq. (4.107) has to equal a constant matrix C. The problem of finding the Kähler potential K_σ, that is covariant under the non-linear symmetries, is reformulated as solving the differential equation

$$(\mathbb{1} + Q) \frac{\mathrm{d}}{\mathrm{d}Q} Q K(Q) = C, \tag{4.108}$$

with the initial condition $K(0) = \mathbb{1}$. Substituting $Q = 0$ in this equation implies that $C = K(0) = \mathbb{1}$ because $K(Q)$ is an analytic function in Q. The solution satisfying the initial condition is given by $K(Q)$ in eq. (4.103). The Kähler potential K_σ for a symmetric coset G/H transforms covariantly under the transformations $\bar{\epsilon}_\alpha \delta^\alpha$ as

$$\bar{\epsilon}_\alpha \delta^\alpha K_\sigma = i \frac{1}{f} \bar{\epsilon} A z. \tag{4.109}$$

4.5.1 Matter Coupling to Symmetric Cosets

We now describe matter coupling to symmetric coset spaces. A covariant vector v^α of a Kähler manifold transforms under the isometries as a differential $\mathrm{d}z^\alpha$, that is $\delta_i v^\alpha = R_{i,\beta}^{\ \alpha} v^\beta$. For later use we give the transformation rules for the non-linear symmetries explicitly

$$[X^\alpha, v^\beta] = f M^{\alpha\beta}_{\gamma\delta} (A^{-1} z)^\gamma v^\delta, \qquad\qquad [\bar{X}_\alpha, v^\beta] = 0. \tag{4.110}$$

Now notice that the transformation rules (4.110) are all linear in v^α by construction. This means that the transformation rules never change the number of v^α's in any term in the Kähler potential. In particular any Kähler potential for covariant vectors has to be invariant. As in the case of the construction of the Kähler potential we notice that for any analytic function G,

$$K_{vec} \;=\; \bar{v} G(Q) v \tag{4.111}$$

is invariant under linear transformations. To obtain the correctly normalized result at low energy, we use the initial condition $G(0) = \mathbb{1}$. We can write

$$\bar{v} Q^n v \;=\; M^{\alpha_1 \alpha_0}_{\beta_1 \delta_1} M^{\alpha_2 \beta_1}_{\beta_2 \delta_2} \ldots M^{\alpha_n \beta_{n-1}}_{\delta_0 \delta_n} \, S^{\delta_0}_{\alpha_0} T^{\delta_n}_{\alpha_1} \ldots T^{\delta_n}_{\alpha_n} \tag{4.112}$$

where $S^\delta_\alpha = \bar{v}_\alpha v^\delta$ transforms under the non-linear symmetries as

$$\bar{\epsilon}_\beta \delta^\beta \left(S^\delta_\alpha\right) = 2\frac{i}{f}\bar{\epsilon}_\alpha \frac{1}{2} f^2 M^{\alpha\gamma}_{\epsilon\phi}(A^{-1}z)^\epsilon v^\phi \bar{v}_\delta \tag{4.113}$$

and T^δ_α transforms according to eq. (4.106) so that

$$\bar{\epsilon}_\beta \delta^\beta \left(\bar{v} Q^n v\right) \;=\; \frac{i}{f}\bar{\epsilon}_\alpha M^{\alpha\beta}_{\gamma\delta}\left(nQ^{n-1} + nQ^n + 2Q^n\right)^\epsilon_\beta (A^{-1}z)^\gamma S^\delta_\epsilon, \tag{4.114}$$

where the symmetry properties of the tensor M have been used. For K_{vec} to be invariant the function G has to obey the differential equation

$$(\mathbb{1} + Q)\frac{\mathrm{d}}{\mathrm{d}Q}G + 2G = 0. \tag{4.115}$$

The solution of this equation that satisfies the initial condition $G(0) = \mathbb{1}$ is given by

$$G(Q) \;=\; (\mathbb{1} + Q)^{-2}. \tag{4.116}$$

Notice that this is precisely the metric which follows from the Kähler potential (4.103).

Next we discuss the coupling of a non-trivial singlet to a symmetric coset space. Let s be invariant under all linear symmetries except the $U(1)$ symmetries under which it transforms with charges y^0_I. Since the commutator of X^α and $\bar{X}_\beta$ is proportional to the charge, it follows that this singlet also must transform under the non-linear symmetries. To obtain the (non-linear) transformation rule for s we take a diagonal H-invariant matrix $y^0 = y_I A_I^{-1}$ such that

$$\delta_i s = (y^0)^\beta_{\ \alpha} R^\alpha_{i,\beta} \tag{4.117}$$

gives the $U(1)$-charges y^0_I. The non-linear transformation rules for the singlet takes the form

$$\bar{\epsilon}_\alpha \delta^\alpha s = i\bar{\epsilon}_\alpha [X^\alpha, s] = f\,\bar{\epsilon} A z\, s \tag{4.118}$$

Since s is a singlet with respect to the semi-simple part of H, the form of the Kähler potential which reduces to the canonical kinetic function in the low energy limit is of the form

$$K_{singlet} = \bar{s} L(\bar{z}, z) s. \tag{4.119}$$

By demanding that $K_{singlet}$ is invariant, we find that

$$L(\bar{z}, z) = e^{-f^2 K_\sigma(\bar{z}, z)}, \tag{4.120}$$

since $\bar{s}s$ transforms proportionally to the Kähler potential K_σ of the coset space (4.109). As this method is purely perturbative we do not see the constraints that the consistency of the line bundle may impose on y^0_I. For this we have to resort to the techniques of subsection 4.3.2.

4.6 Anomalies of supersymmetric G/H cosets

As explained in section 2.4 we try to construct supersymmetric models which are free of isometry anomalies. We now motivate this approach by a discussion of various aspects of supersymmetric G/H models and their anomalies. In particular we focus on symmetric homogeneous Kählerian cosets, as we discuss examples of models based on such cosets in the remaining sections of this thesis. In refs. [95, 34, 94, 33] it is argued that a homogeneous Kählerian coset $\mathrm{G}/\mathrm{H} \cong \mathrm{G}^{\mathbb{C}}/\hat{\mathrm{H}}$ can never arise from a linear G-invariant supersymmetric model by spontaneous symmetry breaking.

When one insists that the underlying theory is renormalizable and linear, one may concluded that supersymmetric models based on Kählerian G/H are not interesting for phenomenology. We feel that renormalizability as such cannot be a constraint when one considers models that are supposed to be valid close to the Planck scale.

Anomalies due to chiral fermions are not a problem specific to supersymmetric models; chiral fermions can be coupled to a coset space G/H in various ways producing anomalies. The chiral super-partners, of a supersymmetric model based on G/H reside in special representations: they are a section of the tangent bundle over G/H. This implies that the target-space connection in the supersymmetric case is fixed. In the non-supersymmetric case has the largest freedom to couple fermions, e.g. using the smallest holonomy group by employing a connection with torsion [60]. The holonomy group $\mathcal{H}$ is in general larger than the isotropy group H, for a supersymmetric model based on the coset space G/H. But they are identical for a symmetric coset [60, 47, 33].

In refs. [60, 120, 121] the conditions for cancellation of isometry anomalies of a σ-model with fermions coupled to it were discussed in detail. Either one adds more fermions such that the fermions form an anomaly-free representation of the isometry group or one needs a Wess-Zumino counter term. Such a counter-term only exist if the 't Hooft's anomaly matching conditions [122] can be satisfied

$$\mathrm{Tr}_{\mathrm{H}}\left(\{T_i, T_j\}T_k\right) = \mathrm{Tr}_{\mathrm{G|H}}\left(\{T_i, T_j\}T_k\right). \tag{4.121}$$

Here Tr_{H} denotes the trace over H representations of the fermions coupled to the σ-model, while $\mathrm{Tr}_{\mathrm{G|H}}$ is the trace over G representations of the preon model restricted to H generators. Using this anomaly matching condition, it can be shown that the supersymmetric models based on the homogeneous coset spaces [60, 33].

$$\begin{array}{cc}
SU(M+N)/[SU(M)\times U(1)\times SU(N)], & E_6/[SO(10)\times U(1)], \\
E_7/[SU(5)\times SU(3)\times U(1)], & E_8/[SO(10)\times SU(3)\times U(1)], \\
SO(2N)/U(N), & Sp(2N)/U(N),
\end{array}$$

are all anomalous. This can be easily seen by looking at the $U(1)$ charges of the coordinates of the cosets.

Apart from these isometry anomalies also the global anomalies, see section 2.4, have been considered in the literature [123, 124, 125, 126]. In ref. [60] is argued that whenever there exists a counter term for the isometry anomalies, the global anomaly vanishes.

4.7 The Sphere and Hyperbolic Space Revisited

In this section we return to our examples of the non-linear Wess-Zumino models, discussed previously in sections 2.2.1, 2.3.1 and 3.4. It is easy to show that these models are examples of Kählerian coset spaces using any of the techniques discussed in sections 4.3, 4.4 and 4.5.

The Kähler potentials (2.37) for these models were obtained in section 2.2.1 by considering the induced metric on a two-sphere or a two-dimensional hyperbolic space. Using the techniques of section 4.4 it is easy to see that these are the Kähler potentials (4.54) of the coset spaces $SU(2)/U(1)$ and $SU(1,1)/U(1)$ for the two-sphere and two-dimensional hyperbolic space respectively. The non-linear transformations (4.47) for these coset spaces are given by

$$ {}^{g}z = (\gamma 2R + \delta z)\left(\alpha + \beta z/(2R)\right)^{-1}, \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU_\eta(1,1), \tag{4.122} $$

where we have introduced the characteristic length-scale R of the sphere or hyperbolic space. The change of coordinate patches described by equation (2.32) is a special case of these transformations

$$ {}^{g}z = \frac{(2R)^2}{z} \quad \text{with} \quad g = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{4.123} $$

This shows that the coordinate transformation for the sphere relating both coordinate patches discussed in 2.2.1 can be viewed as a particular action of the group $SU(2)$. This is a generic property of homogeneous coset spaces: any point of the coset space can be obtained by a group transformation.

As for matter coupling, subsection 4.4.2 provides us with the fundamental matter representation

$$ {}^{g}s = (\delta^{\dashv} - z\beta^{\dashv})^{-1}s = (\alpha + \beta z)^{-1}s, \tag{4.124} $$

using that s is just a complex number. As all $U(1)$ representations are one-dimensional this matter representation is the same as the one obtained as a section of the minimal line bundle. If we take the group element

$$ g = \begin{pmatrix} 1 - i\theta_Y & -\eta\bar{\epsilon} \\ \epsilon & 1 + i\theta_Y \end{pmatrix} \tag{4.125} $$

we can obtain the infinitesimal transformations with $f = (2R)^{-1}$

$$\delta z = \frac{1}{f}\epsilon + 2i\theta_Y z + \eta f z\bar{\epsilon} z, \qquad \delta s = (i\theta_Y + \eta f z\bar{\epsilon})s. \tag{4.126}$$

This coincides with the results of subsection 2.3.1 and 3.4.

We end this subsection with an illustration of anomaly cancellation using matter coupling. The Wess-Zumino models based on the cosets $SU_\eta(1,1)/U(1)$ discussed in 2.3.1 are unsatisfactory as the chiral fermion ψ_L gives rises to $U(1)$ isometry and mixed anomalies. By adding one matter multiplet $\Psi = (x, \chi_L)$ to the model one can remove these anomalies. We take the scalar component s of the matter multiplet Ψ to transform as the derivative $\frac{\partial}{\partial z}$, hence its transformations under the isometries of the coset are given by equation (3.21). Clearly the pure Y- and mixed Y-gravitational anomalies cancel. The kinetic terms for x and χ_L are determined by the matter Kähler potential (3.23) using eq. (2.39).

Chapter 5

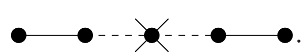
Supersymmetric Grassmannian Models

5.1 Introduction

In the first part of this thesis we have developed the theory of supersymmetric non-linear σ-models in general and those based on Kählerian coset spaces in particular. We now turn to $SU_\eta(M,N)/S[U(M)\times U(N)]$ Grassmannian coset spaces that can be either compact ($\eta = 1$) or non-compact ($\eta = -1$). As we have seen in section 4.4 other coset spaces can be obtained from a Grassmannian coset by embedding them in it. We first give the algebra and the Kähler potential. Next we discuss the infinitesimal transformations. After that we discuss the various options we have to construct matter representations. We describe a supersymmetric σ-model based on the coset $SU_\eta(2,3)/S[U(2)\times U(3)]$ that has the same fermion content as the standard model and which is therefore anomaly-free. We show that it is possible to satisfy the line bundle consistency condition for the compact version. We close with a discussion of the vacuum configuration of both variants of the Grassmannian standard model.

5.2 $SU_\eta(M,N)/S[U(M)\times U(N)]$ Grassmannian Models

A Grassmannian manifold is a homogeneous space that is obtained by considering the coset $SU_\eta(M,N)/S[U(M)\times U(N)]$. The painted Dynkin diagram of this coset is given by



(5.1)

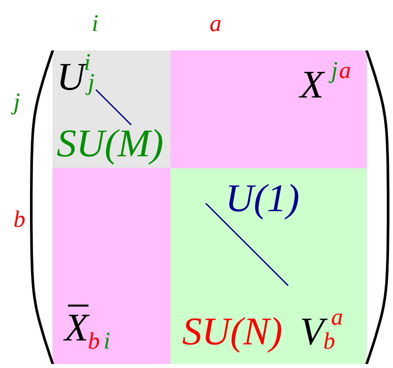
To describe the algebra $SU_\eta(M,N)$ we introduce the indices $i = 1,\dots,M$ and $a = 1,\dots N$ and let $U^k_l, V^c_d, X^{ia}, \bar{X}_{ai}$ and Y be the generators of $SU_\eta(M,N)$. The

U, V and Y are taken to be anti-Hermitean and X and $\bar{X}$ are each others Hermitean conjugates. The generators U^i_j span the subalgebra $SU(M)$ of $SU_\eta(M,N)$ and similarly the generators V^a_b span the subalgebra $SU(N)$ of $SU_\eta(M,N)$ and Y is the $U(1)$ generator in $SU_\eta(M,N)$ that commutes with all generators U, V of $SU(M)\times SU(N)$. The algebra of $SU_\eta(M,N)$ can be represented as

$$\begin{aligned}
&[Y, X^{ia}] = (M+N)X^{ia}, && [Y, \bar{X}_{ai}] = -(M+N)\bar{X}_{ai},\\
&[U^k_l, X^{ia}] = \delta^i_l X^{ka} - \frac{1}{M}\delta^k_l X^{ia}, && [U^k_l, \bar{X}_{ai}] = -\delta^k_i \bar{X}_{al} + \frac{1}{M}\delta^k_l \bar{X}_{ai},\\
&[V^c_d, X^{ia}] = -\delta^a_d X^{ic} + \frac{1}{N}\delta^c_d X^{ia}, && [V^c_d, \bar{X}_{ai}] = \delta^c_a \bar{X}_{di} - \frac{1}{N}\delta^c_d \bar{X}_{ai},\\
&[U^i_j, U^k_l] = \delta^k_j U^i_l - \delta^i_l U^k_j, && [V^a_b, V^c_d] = \delta^a_d V^c_b - \delta^c_b V^a_d,
\end{aligned} \tag{5.2}$$

$$[\bar{X}_{ai}, X^{jb}] = \eta\left(\delta^b_a U^j_i - \delta^j_i V^b_a\right) + \eta\frac{1}{MN} Y \delta^j_i \delta^b_a.$$

The different generators can be schematically represented by the picture



The non-linear realization of the $U_\eta(M,N)$ algebra on multiplets Q and $\bar{Q}$ takes the form

$$\begin{aligned}
\delta Q = R(Q) &= \tfrac{1}{f}\epsilon + \eta f Q\bar{\epsilon}Q + iuQ - iQv + i(M+N)\theta Q,\\
\delta\bar{Q} = \bar{R}(\bar{Q}) &= \tfrac{1}{f}\bar{\epsilon} + \eta f \bar{Q}\epsilon\bar{Q} + iv\bar{Q} - i\bar{Q}u - i(M+N)\theta\bar{Q},
\end{aligned} \tag{5.3}$$

where u is an element of $SU(M)$, v of $SU(N)$ and ϵ is an $M\times N$-matrix and $\bar{\epsilon}$ an $N\times M$-matrix. The $U(1)$ symmetry is parameterized by θ. The parameter f has the dimension of inverse mass. It therefore sets the scale of the σ-model and gives the fields Q the canonical dimension. The superfield matrix $Q = (Q^{ia})$ has vector indices in both $SU(M)$ and $SU(N)$ and $\bar{Q} = (\bar{Q}_{ai})$ is its conjugate. In subsection 5.3 we shall interpret Q^{ia} as a chiral multiplet containing a quark-doublet. In the conventions used here Q^{ia} has a Y-charge $M+N$. These transformation rules can be obtained from the non-linear transformation (4.47) or by the method described in section 4.5. On Q^{ia} the generators U and V act via left and right multiplication, respectively. For this reason the commutators involving V differ from the commutators containing U by a minus sign in (5.2).

The inverse metrics

$$\left(\chi^{-1}\right)^i_{\ j} \;=\; \left[1+\eta f^2 Q\bar{Q}\right]^i_{\ j}, \qquad \left(\tilde{\chi}^{-1}\right)^a_{\ b} \;=\; \left[1+\eta f^2 \bar{Q}Q\right]^a_{\ b}. \tag{5.4}$$

transform under these symmetries as

$$\delta\chi^{-1} = H\chi^{-1} + \chi^{-1}H^\dagger, \qquad \delta\tilde{\chi}^{-1} = \tilde{\chi}^{-1}\tilde{H} + \tilde{H}^\dagger\tilde{\chi}^{-1}. \tag{5.5}$$

The holomorphic matrix-valued functions given by

$$H = \eta f Q\bar{\epsilon} + iu + iN\theta \quad \text{and} \quad \tilde{H} = \eta f \bar{\epsilon} Q - iv + iM\theta \tag{5.6}$$

and their conjugates form a non-linear representation of $SU_\eta(M,N)$. The Kähler potential [50, 51, 52] for the Grassmannian σ-models can be written as

$$K_\sigma(\bar{Q},Q) \;=\; \frac{1}{\eta f^2}\mathrm{tr}_M \ln(\chi^{-1}) \;=\; \frac{1}{\eta f^2}\mathrm{tr}_N \ln(\tilde{\chi}^{-1}). \tag{5.7}$$

The inverse metrics χ^{-1} and $\tilde{\chi}^{-1}$ are given in eq. (5.4). Two traces have been introduced: tr_M acts on $M\times M$-matrices and tr_N on $N\times N$-matrices. This Kähler potential is directly obtained from (4.64). Using (5.5) it is easy to show that $K_\sigma(\bar{Q},Q)$ in eq. (5.7) transforms as a Kähler potential

$$\delta K_\sigma(\bar{Q},Q) = F(Q) + \bar{F}(Q). \tag{5.8}$$

The holomorphic function F

$$F(Q) = \frac{1}{\eta f^2}\mathrm{tr}_M H = \frac{1}{\eta f^2}\mathrm{tr}_N \tilde{H} = \frac{1}{\eta f^2}\left(\eta f \mathrm{tr}_M(Q\bar{\epsilon}) + iMN\theta\right) \tag{5.9}$$

also form a non-linear representation, as the functions H and $\tilde{H}$ do.

Next we discuss matter coupling to the Grassmannian model. Let $\mathfrak{R}(\bar{\Sigma},\Sigma)$ and $\tilde{\mathfrak{R}}(\bar{\Sigma},\Sigma)$ be $M\times M$- and $N\times N$-matrix-valued composite real superfields, respectively. They are called *left* and *right* covariant, respectively, if they transform as

$$\delta\mathfrak{R} = H\mathfrak{R} + \mathfrak{R}H^\dagger, \qquad \delta\tilde{\mathfrak{R}} = \tilde{\mathfrak{R}}\tilde{H} + \tilde{H}^\dagger\tilde{\mathfrak{R}} \tag{5.10}$$

under the $SU_\eta(M,N)$ isometries of the Grassmannian manifold. Invariant Kähler potentials for these real composite superfields $\mathfrak{R}$ and $\tilde{\mathfrak{R}}$ are provided by

$$\mathrm{tr}_M(\chi\mathfrak{R}), \qquad \mathrm{tr}_N(\tilde{\chi}\tilde{\mathfrak{R}}). \tag{5.11}$$

Consider the chiral multiplets L^i and $\tilde{D}^a$ which transform under $SU_\eta(M,N)$ by left, resp. right multiplication

$$\begin{aligned} \delta L &= HL = (\eta f Q\bar{\epsilon} + iu + iN\theta)L, \\ \delta\tilde{D} &= \tilde{D}\tilde{H} = \tilde{D}(\eta f\bar{\epsilon}Q - iv + iM\theta). \end{aligned} \tag{5.12}$$

The charges of L and $\tilde{D}$ are N and M, respectively. We will later, in section 5.3, interpret L and Q as chiral superfields containing the left-handed lepton doublets and left-handed quark doublets. We also want to obtain matter representations that can be used as the superfields containing the charge conjugates of the u and d quarks. For this we cannot use $\tilde{D}$ as it transforms in the $\underline{3}$ and not in the $\underline{\bar{3}}$. We obtain a $\underline{\bar{3}}$ representation of $SU(3)$ if we take a superfield D that transforms as

$$\delta D = -\tilde{H}D = -(\eta f\bar{\epsilon}Q - iv + iM\theta)D. \tag{5.13}$$

Its charge is $-M$. However this interpretation still does not work directly as the charges of L and D with respect to Q turn out not to be compatible with the standard model hyper charges.

Notice that $(L\bar{L})_i^{\ j}$ and $(\tilde{D}^\dagger\tilde{D})_a^{\ b}$ are left and right covariant composite superfields, respectively. Hence by eqs. (5.11) Kähler invariants can be constructed of the form

$$\bar{L}\chi L \qquad \text{and} \qquad \tilde{D}\tilde{\chi}\tilde{D}^\dagger. \tag{5.14}$$

By taking tensor products of multiplets which transform like L and D, one can obtain (p,q)-rank $U(M)\times U(N)$-tensor chiral multiplet with charge $pN+qM$. In particular the $(1,1)$-rank tensor multiplet T^{ia} has the same charge as Q and it transforms as the differentials dQ; this is an example of the construction given in eq. (3.10). In this way the metric of the Grassmann manifold can be obtained

$$g_\sigma{}^{ai}{}_{jb} = \frac{\partial K}{\partial\bar{Q}_{ai}\partial Q^{jb}} = \chi^i{}_j\tilde{\chi}^a{}_b. \tag{5.15}$$

The Kähler invariant for T^{ia} reads

$$\bar{T}_{ai}g_\sigma{}^{ai}{}_{jb}T^{jb} = \mathrm{tr}_M(\chi T\tilde{\chi}\bar{T}) = \mathrm{tr}_N(\tilde{\chi}\bar{T}\chi T). \tag{5.16}$$

For D we obtain the Kähler invariant

$$\bar{D}\tilde{\chi}^{-1}D. \tag{5.17}$$

Because of the transformation property (5.9) of the function F, we can use (3.14) to couple a $SU(M)\times SU(N)$-singlet multiplet Ω to a Grassmannian manifold. It transforms as

$$\delta\Omega = \eta f^2 F(Q)\Omega = (\eta f\,\mathrm{tr}_M(Q\bar{\epsilon}) + iMN\theta)\,\Omega. \tag{5.18}$$

The charge of this non-trivial singlet is MN. For later convenience we take Ω dimensionless.

Any given chiral multiplet, for example L, can be rescaled by such a (non-physical) singlet Ω to $L' = \Omega^l L$, which transform as

$$\begin{aligned}\delta L' &= \left(l\,\eta f^2 F + H\right) L' \\ &= \left(\eta f Q\bar{\epsilon} + iu + l\,\eta f \mathrm{tr}_M(Q\bar{\epsilon}) + i(l\,MN + N)\theta\right) L'\end{aligned} \tag{5.19}$$

using the transformation (5.18) of the singlet Ω. The charge of L' is equal to $l\,MN + N$. The additional terms in the transformation rule for L' have to be compensated in the Kähler potential for it still to be invariant. Again let $\mathfrak{R}(\bar{\Sigma},\Sigma)$ and $\tilde{\mathfrak{R}}(\bar{\Sigma},\Sigma)$ be left and right covariant real composite multiplets. A left covariant composite real superfield is constructed for L' by $e^{-l\mathrm{tr}_M \ln \mathfrak{R}} L'\bar{L}'$ or $e^{-l\mathrm{tr}_N \ln \tilde{\mathfrak{R}}} L'\bar{L}'$ and hence Kähler invariants for L' are obtained by (5.11). By eqs.(5.5) it follows that χ^{-1} and $\tilde{\chi}^{-1}$ are left and right covariant, respectively. If one takes χ^{-1} and $\tilde{\chi}^{-1}$ for the composite superfields $\mathfrak{R}$ and $\tilde{\mathfrak{R}}$ then one obtains

$$\bar{L}'\chi L'\, e^{-l\eta f^2 K_\sigma} \tag{5.20}$$

by eq. (5.7). This is an example of the general construction discussed in eq. (3.19). From now on we omit the primes. Of course a similar construction works for D as well. The generalization of the Kähler invariants (5.14) when L has been rescaled by l and D by d:

$$\delta L = (H + l\eta f^2 F)L, \qquad \delta D = (-\tilde{H} + d\eta f^2 F)D. \tag{5.21}$$

are given by

$$K_L = \bar{L}\chi^{(L)}L, \qquad K_D = \bar{D}\tilde{\chi}^{(D)}D, \tag{5.22}$$

with the modified metrics

$$\chi^{(L)} = e^{-l\eta f^2 K_\sigma}\,\chi, \qquad \tilde{\chi}^{(D)} = e^{-d\eta f^2 K_\sigma}\,\tilde{\chi}^{-1}. \tag{5.23}$$

We determine the Killing potentials that are needed if we gauge part of the isometries. The non-vanishing $SU_\eta(M,N)$ Killing metric entries are

$$\begin{aligned}\eta_{U^i_j U^k_l} &= -(M+N)\left(\delta^i_l\delta^k_j - \frac{1}{M}\delta^i_j\delta^k_l\right), \quad \eta_{YY} = -MN(M+N)^2, \\ \eta_{V^a_b V^c_d} &= -(M+N)\left(\delta^a_d\delta^c_b - \frac{1}{N}\delta^a_b\delta^c_d\right), \quad \eta_{\bar{X}_{ai}X^{jb}} = \eta(M+N)\delta^j_i\delta^b_a.\end{aligned} \tag{5.24}$$

Next we discuss the Killing potentials. We denote all Killing potentials $\mathcal{M}_i$ collectively by $\mathcal{M} = \theta^i\mathcal{M}_i$, combining the parameters of the $SU_\eta(M,N)$ isometries

$\theta^i = (u, v, \theta, \epsilon, \bar{\epsilon})$. We first focus on the Killing potential $\mathcal{M}_\sigma$ depending on the σ-model fields Q and $\bar{Q}$ only; afterwards the matter contribution $\mathcal{M}_m$ is examined. The complete Killing potential is given by $\mathcal{M} = \mathcal{M}_\sigma + \mathcal{M}_m$. Both the σ-model and matter Killing potentials can be written conveniently in terms of the matrices

$$\begin{aligned}\Delta &= R^{(ia)} \left(\chi^{-1}\right)_{,(ia)} \chi - H \\ &= i\theta \left((M+N)\eta f^2 Q\bar{Q}\chi - N\right) - iu\chi - i\eta f^2 Qv\bar{Q}\chi + \eta f \left(\epsilon\bar{Q} - Q\bar{\epsilon}\right)\chi, \\ \tilde{\Delta} &= \tilde{\chi}\left(\tilde{\chi}^{-1}\right)_{,(ia)} R^{(ia)} - \tilde{H} \\ &= i\theta\left((M+N)\eta f^2 \tilde{\chi}\bar{Q}Q - M\right) + i\eta f^2 \tilde{\chi}\bar{Q}uQ + i\tilde{\chi}v + \eta f\tilde{\chi}\left(\bar{Q}\epsilon - \bar{\epsilon}Q\right)\end{aligned} \tag{5.25}$$

Here we have used the index notation (ia) to emphasize that this index refers to the superfield Q^{ia}. Using eq. (2.17) we find that the Killing potential $\mathcal{M}_\sigma$ can be written as

$$i\mathcal{M}_\sigma = K_{\sigma,(ia)}R^{(ia)} - \frac{1}{q\eta f^2}\mathcal{W}^{-1}\delta\mathcal{W} = \frac{1}{\eta f^2}\mathrm{tr}_M\Delta = \frac{1}{\eta f^2}\mathrm{tr}_N\tilde{\Delta}. \tag{5.26}$$

Here we have assumed that the superpotential transforms covariantly (3.7), hence $\mathcal{W}^{-1}\delta\mathcal{W}$ plays the role of F.

To discuss the Killing potential due to matter fields in some generality we introduce some further notation. We discuss only the rescaled matter field L here, as it is easy to generalize our discussion to the matter field D and tensor products. Define the $M \times M$-matrix composite real-superfield $[L\bar{L}]_i^j = (\chi^{(L)}\, L)^j \bar{L}_i$ where $\chi^{(L)}$ is the rescaled metric defined in eq. (5.23). Notice that $[L\bar{L}]\chi^{-1}$ is a left covariant real composite superfield, hence by (5.11) we obtain the Kähler invariant: $\mathrm{tr}_M[L\bar{L}] = K_L$. From now on we assume that the matter Kähler potential K_m can be written entirely in terms of matrices like $[L\bar{L}]$. As K_m is an invariant Kähler function, one can define the Killing potential for the matter field L as

$$i\mathcal{M}_L = \mathrm{tr}_M\left[K_{m,[L\bar{L}]}\Big(\delta Q^{ia}(\chi^{(L)})_{,(ia)}L\bar{L} + \chi^{(L)}(\delta L)\bar{L}\Big)\right]. \tag{5.27}$$

where $K_{m,[L\bar{L}]}$ denotes the derivative of K_m with respect to the matrix $[L\bar{L}]$. This can be expressed in terms of Δ and $\mathcal{M}_\sigma$ as

$$i\mathcal{M}_L = -\bar{L}K_{m,[L\bar{L}]}\chi^{(L)}\left(l\eta f^2\, i\mathcal{M}_\sigma + \Delta\right)L. \tag{5.28}$$

The Killing potential $\mathcal{M}_m$ due to all the different matter fields is a sum of Killing potentials like $\mathcal{M}_L$. As the Killing potentials $\mathcal{M}_\sigma$, $\mathcal{M}_m$ for the σ-model fields and the matter fields are linear in Δ and $\tilde{\Delta}$, cf. eq. (5.26), we can express the full Killing potential as

$$i\mathcal{M} = \mathrm{tr}_M\Delta P + \mathrm{tr}_N\tilde{\Delta}\tilde{P} \tag{5.29}$$

where the field dependent matrices P and $\tilde{P}$ encode the details of the full Kähler potential. In particular for $\mathcal{M}_\sigma$ we may choose $P_\sigma = \frac{1}{2}\mathbb{1}_M$ an $\tilde{P}_\sigma = \frac{1}{2}\mathbb{1}_N$ while for $\mathcal{M}_L$ we find

$$P_L = -l\eta f^2 i\,\bar{L}K_{m,[L\bar{L}]}g^{(L)}L\,\mathbb{1} - L\bar{L}K_{m,[L\bar{L}]}g^{(L)}. \tag{5.30}$$

Using these matrices P one can state the Killing potentials for the different symmetries of $U_\eta(M,N)$ as

$$\begin{aligned}\mathcal{M}_Y &= \mathrm{tr}_M P\left((M+N)\eta f^2 Q\bar{Q}\chi - N\right) + \mathrm{tr}_N \tilde{P}\left((M+N)\eta f^2 \tilde{\chi}\bar{Q}Q - M\right),\\ \mathcal{M}_U &= -\chi P + \eta f^2 Q\tilde{P}\tilde{\chi}\bar{Q}, \qquad \mathcal{M}_V = \tilde{P}\tilde{\chi} - \eta f^2 \bar{Q}\chi P Q,\\ i\mathcal{M}_X &= \eta f\left(\bar{Q}\chi P + \tilde{P}\tilde{\chi}\bar{Q}\right), \qquad i\mathcal{M}_{\bar{X}} = -\eta f\left(\chi P Q + Q\tilde{P}\tilde{\chi}\right).\end{aligned} \tag{5.31}$$

5.3 Grassmannian standard model

We now illustrate how one can cancel isometry anomalies by adding rescaled matter multiplets. If we consider the case with $M = 2$ and $N = 3$ then the Grassmannian manifold may be the basis of an $SU(5)$ unified model with the standard model group $SU(2)\times U(1)\times SU(3)$ as the unbroken subgroup [69]. We do not require the $SU_\eta(2,3)$ to be compact. In the standard model the field content is such that all possible anomalies cancel in each generation, consisting of a quark doublet q_L, a lepton doublet l_L, quark singlets d_L^c and u_L^c and an electron singlet e_L^c. The notions singlet and doublet here refer to $SU(2)$ representations and the superscript c denotes charge conjugation. In this model only part of the quarks and leptons are superpartners of Goldstone bosons, the coordinates of the coset. If one wants to consider models with more generations, the simplest thing is just to take a number of copies of this structure. Only in the quark doublet sector there will be a difference: an additional quark doublet Q' is to be coupled covariantly to the σ-model. We do not pursue multiple generation models here. Finally we introduce a Higgs sector consisting of two $SU(2)$ doublets $H^\pm$ with opposite charge. The introduction of the Higgses is necessary for the breaking of $SU(2)\times U_Y(1)$ to $U_{em}(1)$.

The hyper-charges in the standard model are assigned such that anomalies cancel. In the supersymmetric models the chiral fermion representations have to be completed to the chiral supermultiplets Q^{ia}, L^i, D_a, U_a, $H^\pm$ and E. However if we use the standard coupling of matter multiplets (5.12) and (5.13) to the Grassmann σ-model we do not obtain the hypercharge assignment of the standard model, see table 5.1. There the hypercharge Y_w and the canonical charge Y for various standard model multiplets are given, they are not proportional with one common proportionality factor. However from eq. (5.18) we see that the singlet chiral multiplet Ω has $U(1)$ charge $MN = 6$ in the $SU_\eta(2,3)$ model. By

Multiplet	Fermion	$SU(2)\times SU(3)$	Y	Y_w	k
Q^{ia}	q_L^{ia}	(2,3)	5	+1/3	0
Q'^{ia}	$q_L'^{ia}$	(2,3)	5	+1/3	0
L^i	l_L^i	(2,1)	3	-1	$l=-3$
H^{-i}	h_L^{-i}	(2,1)	3	-1	$h^-=-3$
H^{+i}	h_L^{+i}	(2,1)	3	+1	$h^+=+2$
D_a	$d_{L\,a}^c$	$(1,\bar{3})$	-2	+2/3	$d=+2$
U_a	$u_{L\,a}^c$	$(1,\bar{3})$	-2	-4/3	$u=-3$
E	e_L^c	(1,1)	0	2	$e=5$
Ω	-	(1,1)	6	-	-

Table 5.1: Grassmannian (matter) multiplets and their chiral fermion content classified by their $SU(2)\times SU(3)$ properties. Y is the canonical charge of the σ model and Y_w denotes the hypercharge needed for anomaly cancellation within the standard model. These charges can be identified if $Y = 15Y_w$. The number k gives the rescaling-factors with a singlet Ω.

employing the rescaling: $\Psi^{(k)} = \Omega^k\Psi$ any chiral multiplet Ψ can be given an additional charge $k\,MN$. Therefore we find the relation

$$Y + 6\,k = \lambda Y_w. \tag{5.32}$$

As the coordinates Q^{ia} have charge $Y = 5$ and cannot have a rescaling charge ($k = 0$), we find that $\lambda = 15$. In the last column we have given the powers ($k = l, d, u, e, h^\pm$) to which the singlet has to be raised in order the find the right hypercharge assignment for the standard model. We know that for compact Grassmannian cosets the minimal charge of the line bundle is equal to MN, as was discussed from the general discussion of matter coupling to coset spaces, in sections 4.3.2 and 4.4.2. The last column of table 5.1 shows that we only need integer powers of Ω. Therefore the line bundle cocycle condition is satisfied for each of these matter representations.

In the following we assume that we have performed the rescaling to the chiral multiplets as given in this table and hence we can state the Kähler potential:

$$K \;=\; K_\sigma + K_E + K_L + K_D + K_U + K_{H^+} + K_{H^-}, \tag{5.33}$$

where K_L and K_D are defined in eqs. (5.22) and K_E, $K_{H^\pm}$ and K_U are defined in a similar fashion.

As fundamental compensating superpotentials we may take

$$\begin{aligned} w_E &= fE & &(q_E = e = 5), \\ w_{L-} &= f^2\varepsilon_{ij}L^iH^{-j} & &(q_{L-} = 1 + l + h^- = -5). \end{aligned} \tag{5.34}$$

These compensating superpotentials transform as a non-trivial singlet given by $\delta w = q\eta f^2 F\, w = q\, \mathrm{tr} H\, w$, where the numbers q are given in the brackets in (5.34). The numbers e, l and h^- can be read off from table 5.1. As the contraction with the Levi-Cevita ε-tensor gives a $SU(2)$ but not a $U(2)$ invariant, it follows that the transformation of w_{L-} includes an additional contribution of $\mathrm{tr}H$. For the invariant superpotential W we take a part of the standard model superpotential:

$$W = \alpha + \beta E \varepsilon_{ij} H^{-i} L^j - \mu \varepsilon_{ij} H^{+i} H^{-j}. \tag{5.35}$$

The first term α is a constant with dimension of $(\text{mass})^3$. The second term is the usual Yukawa coupling in supersymmetric models and the third term is the Higgs interaction. Notice that in this model there are no Yukawa interactions for the quarks, as the quark doublet superfield Q does not transform covariantly. Using w_E and w_{L-} we obtain the covariant superpotentials

$$\mathcal{W}_E = w_E W, \qquad \mathcal{W}_{L-} = w_{L-} W, \tag{5.36}$$

which have the same transformation properties as w_E and w_{L-}. We restrict ourselves here to these two examples of covariant superpotentials to illustrate the situation, but one could construct many more covariant superpotentials by taking integral powers of w_E and w_{L-}.

If chiral multiplets are described by a covariant Kähler and a covariant superpotential in supergravity, this implies the relation $\kappa^2 = -\eta f^2\, q$ between Newton's constant and the σ-model scale, see subsection 3.5.2. As Newton's constant and the σ-model scale are positive, we see that the signs of η and q have to be opposite from this relation between κ and f. The superpotential $\mathcal{W}_E$ is therefore compatible with the non-compact version of the Grassmannian standard model while $\mathcal{W}_{L-}$ is compatible with the compact Grassmannian standard model.

5.4 Geometrical objects for matter

Section 3.6 was devoted to the question of how one could make the metric of the combined system of matter and σ-model fields block-diagonal. In this section we illustrate how some of methods discussed there work in practice with the example discussed in section 5.3 of consistent Grassmann σ-models with the field content of the standard model with one generation. Our starting point is the quadratically coupled matter Kähler potential (5.33). Using the results of section 3.6 we have computed the connections (3.50)

$$\begin{aligned}
\Gamma^{(kc)}_{(ia)(jb)} &= -\,\eta f^2 \left(\delta^c_b \delta^k_i (\tilde{\chi}\bar{Q})_{aj} + \delta^c_a \delta^k_j (\tilde{\chi}\bar{Q})_{bi}\right), \\
\Gamma_{E(jb)} &= -\,e\eta f^2 (\bar{Q}\chi)_{jb} E, \\
\Gamma^k_{L(jb)} &= -\,\eta f^2 \left(l(\bar{Q}\chi)_{bj} L^k + \delta^k_j (\bar{Q}\chi L)_b\right), \\
\Gamma^c_{\tilde{D}(jb)} &= -\,\eta f^2 \left(d(\tilde{\chi}\bar{Q})_{bj} \tilde{D}^c + \delta^c_b (\tilde{D}\tilde{\chi}\bar{Q})_j\right), \\
\Gamma_{Dc\,(jb)} &= -\,\eta f^2 \left(d(\tilde{\chi}\bar{Q})_{bj} D_c - (\tilde{\chi}\bar{Q})_{cj} D_b\right).
\end{aligned} \tag{5.37}$$

The connection for U is similar to the one for D, and the connections for the Higgses $H^{\pm}$ are similar to the one for L. To make a distinction between indices referring to the original σ-model fields Q^{ia} and matter indices a and i, we write (ai) for the former ones. Notice that the normal gauge, in which all connections vanish, coincides with the unitary gauge $Q = 0$. Because of the global $SU_\eta(2,3)$ invariance, the vacuum can always be studied in the normal gauge by setting $\langle Q \rangle = 0$. Using these connections, one obtains the covariant chiral fermions of eq. (3.58), for example

$$\begin{aligned}
\hat{e}^c_L \equiv e^{c\prime}_L &= e^c_L - e\eta f^2 E \;\mathrm{tr}_M \bar{Q}\chi q_L, \\
\hat{l}^i_L \equiv l^{i\prime}_L &= l^i_L - \eta f^2 \left(l L^i \;\mathrm{tr}_M \bar{Q}\chi q_L + (q_L \bar{Q}\chi L)^i\right).
\end{aligned} \tag{5.38}$$

Because we only consider quadratically coupled matter here, we find that the connections $\Gamma^A_{x\alpha} = x^B \Gamma^A_{B\alpha}$ and $\Gamma^A_{BC} = 0$. For the same reason most of the curvatures of eq.(3.54) are related. In particular we find

$$\begin{aligned}
R^{(ia)}{}_{(bj)}{}^{(kc)}{}_{(dl)} &= -\,\eta f^2 \left(g_\sigma{}^{(ic)}{}_{(bj)}\, g_\sigma{}^{(ka)}{}_{(dl)} + g_\sigma{}^{(ka)}{}_{(bj)}\, g_\sigma{}^{(ic)}{}_{(dl)}\right), \\
R_{\bar{E}E}{}^{(bj)}{}_{(ia)} &= -\,\eta f^2 e K_E \chi^j{}_i \tilde{\chi}_a{}^b \;= -\eta e f^2 K_E g_\sigma{}^{(bj)}{}_{(ia)} \\
R_{\bar{L}L}{}^{(bj)}{}_{(ia)} &= -\,\eta f^2 \left(l K_L \chi^j{}_i + ([L\bar{L}]\,\chi)^j_i\right) \tilde{\chi}_a{}^b, \\
R_{\bar{\tilde{D}}\tilde{D}}{}^{(bj)}{}_{(ia)} &= -\,\eta f^2 \left(d K_{\tilde{D}} \tilde{\chi}_a{}^b + (\tilde{\chi}\,[\tilde{D}^\dagger \tilde{D}])^b_a\right) \chi^j{}_i, \\
R_{\bar{D}D}{}^{(bj)}{}_{(ia)} &= -\,\eta f^2 \left(d K_D \tilde{\chi}_a{}^b - (\tilde{\chi}\,[D\bar{D}])^b_a\right) \chi^j{}_i,
\end{aligned} \tag{5.39}$$

using the notation $[L\bar{L}]$, etc., of section 5.2. The metric $G_\sigma{}^{(bj)}{}_{(ia)}$ of the σ-model fields Q and $\bar{Q}$ in the presence of matter multiplets E, L, D, U becomes

$$G_\sigma{}^{(bj)}{}_{(ia)} \equiv g_\sigma{}^{(bj)}{}_{(ia)} + \sum_x R_{\bar{x}x}{}^{(bj)}{}_{(ia)} = \alpha \left(\chi \otimes \tilde{\chi} + \chi A \otimes \tilde{\chi} + \chi \otimes B\tilde{\chi}\right)^{(bj)}{}_{(ia)}, \tag{5.40}$$

using eq.(3.52) as well as the curvatures (5.39) with the short-hand notations

$$\begin{aligned} \alpha = 1 - \eta f^2 \textstyle\sum_x q_x K_x, \qquad B = -\eta f^2 \alpha^{-1} \left([D\bar{D}] + [U\bar{U}] \right), \\ A = -\eta f^2 \alpha^{-1} \left([L\bar{L}] + [H^+\bar{H}^+] + [H^-\bar{H}^-] \right). \end{aligned} \tag{5.41}$$

Notice that in the unitary gauge $Q = 0$ the metric G_σ does not reduce to the metric without matter coupling g_σ evaluated at $Q = 0$. The inverse of this metric can be written as infinite sum of tensor products

$$G_\sigma^{-1} = \alpha^{-1} \sum_{n=0}^{\infty} (1 + A)^{-n-1} A^n \chi^{-1} \otimes \tilde{\chi}^{-1} B^n (1 + B)^{-n-1}. \tag{5.42}$$

The combined σ-model and matter metric in the transformed system is given by

$$G' = \mathrm{diag} \left(G_\sigma, \chi^{(E)}, \chi^{(L)}, \chi^{(H^+)}, \chi^{(H^-)}, \tilde{\chi}^{(D)}, \tilde{\chi}^{(U)} \right), \tag{5.43}$$

where G_σ is given by (5.40).

5.5 Grassmannian standard model vacuum

In section 5.3 we presented anomaly-free Grassmannian models with the fermion particle spectrum identical to the standard model. We now discuss the possible vacuum configurations of these models. Grassmannian models with doubling have been studied in a supergravity background [118, 127]. We focus on the Grassmannian standard models described previously and include a superpotential.

Before going into the details of the model we first observe that locally the vacuum can always be chosen such that $\langle Q \rangle = 0$. As the vacuum expectation values of Q and $\bar{Q}$ are constants, they can be set to zero by a global gauge transformation. Notice that $\langle Q \rangle = 0$ is indeed a vacuum solution, because in the scalar potential Q and $\bar{Q}$ always appear together.

In the supergravity background the model of $SU_\eta(2,3)/[SU(2) \times U(1) \times SU(3)]$ with the chiral fermion content of the standard model should satisfy at least the following requirements in order not to be in conflict with the standard model phenomenology: the gauge group $SU(3) \times U_{em}(1)$ is unbroken, and the gauginos and the complex scalar bosons should acquire masses above the scale of the gauge bosons and the chiral fermions.

Here we only analyze the restrictions resulting from the electroweak symmetry breaking. The subgroup $SU(3) \times U_Y(1) \times SU(2)$ is gauged and the generator $Q_{em} = \frac{1}{2} Y_w + I_3$ is unbroken. Therefore all $SU(2) \times SU(3)$-singlets under should vanish in the vacuum. In particular, the covariant superpotential $\mathcal{W}$ should have a zero vacuum expectation value $\langle\mathcal{W}\rangle = 0$. Only neutral components of the Higgs $SU(2)$ doublets may acquire a vacuum expectation value

$$\langle H^+\rangle = \begin{pmatrix} 0 \\ H_0^+ \end{pmatrix}, \qquad \langle H^-\rangle = \begin{pmatrix} H_0^- \\ 0 \end{pmatrix}. \tag{5.44}$$

The Killing potentials (5.31) of the Y-charge and the weak-isospin

$$\begin{aligned} \mathcal{M}_Y &= -\frac{6}{\eta f^2} + 15\left(|H_0^+|^2 - |H_0^-|^2\right), \\ \mathcal{M}_{I_3} &= \frac{1}{2}\left(|H_0^-|^2 - |H_0^+|^2\right) \end{aligned} \tag{5.45}$$

are the only Killing potentials which do not necessarily vanish. The non-vanishing part of the scalar potential due to the D-terms is given by

$$V_D = \frac{1}{2}g_Y^2\mathcal{M}_Y^2 + \frac{1}{2}g_2^2\mathcal{M}_{I_3}^2. \tag{5.46}$$

We denote the $U(1)$ gauge coupling constant by g_Y, the gauge coupling constants for $SU(2)$ and $SU(3)$ by g_2 and g_3, respectively. We observe that there always is a D-term supersymmetry and internal symmetry breaking, and the minimum of the potential occurs at

$$|H_0^+|^2 - |H_0^-|^2 = -2\mathcal{M}_{I_3} = \frac{15g_Y^2}{(15g_Y)^2 + \frac{1}{4}g_2^2}\left(\frac{6}{\eta f^2}\right). \tag{5.47}$$

The other Killing potential takes the value

$$\mathcal{M}_Y = \frac{-\frac{1}{4}g_2^2}{(15g_Y)^2 + \frac{1}{4}g_2^2}\left(\frac{6}{\eta f^2}\right). \tag{5.48}$$

To analyze the F-term part of the scalar potential we have to specify which superpotential ($\mathcal{W}_E$ or $\mathcal{W}_{L-}$) we are using. We start our analysis of the different cases with $\mathcal{W}_E$. Because the covariant superpotential vanishes $\langle\mathcal{W}\rangle = 0$, the minimum of the potential is given by $\langle\mathcal{W}_{E,\mathcal{A}}\rangle = 0$, we find for $\mathcal{A} = E$

$$\alpha - \mu|H_0^+||H_0^-| = 0. \tag{5.49}$$

From this equation together with (5.47) we get a prediction for the ratio of the two vacuum expectation values of the Higgses

$$\tan^2\beta \equiv \frac{|H_0^+|^2}{|H_0^-|^2} = \frac{\sqrt{\mathcal{M}_{I_3}^2 + (\alpha/\mu)^2} - \mathcal{M}_{I_3}}{\sqrt{\mathcal{M}_{I_3}^2 + (\alpha/\mu)^2} + \mathcal{M}_{I_3}}, \tag{5.50}$$

where $\mathcal{M}_{I_3}$ is given by eq. (5.47).

Finally we consider the case of $\mathcal{W} = \mathcal{W}_{L-}$. The vanishing of the derivatives of the covariant superpotential demands that either $H_0^- = 0$ or eq. (5.49) is satisfied. There are two inequivalent vacua which both break the electroweak symmetry. First of all

$$H_0^- = 0, \quad |H_0^+|^2 = -2\mathcal{M}_{I_3} = \frac{15 g_Y^2}{(15 g_Y)^2 + \frac{1}{4} g_2^2} \left(\frac{6}{\eta f^2} \right), \tag{5.51}$$

which gives the unacceptable result $\tan\beta = \infty$. The other vacuum solution leads to a $\tan\beta$ as given in eq. (5.50).

Chapter 6

Supersymmetric $SO(2N)/U(N)$ Models

6.1 Introduction

In this chapter we discuss the Kähler coset space $SO(2N)/U(N)$ and its applications to supersymmetric model building. We start by reviewing the decomposition of the $SO(2N)$ algebra in $U(N)$ representations. We discuss the vector and spinor representations of $SO(2N)$ in more detail. The spinor representation is decomposed into completely anti-symmetric $SU(N)$ representations. Using the vector representation we can construct the Kähler potential of $SO(2N)/U(N)$ and give a general discussion of anti-symmetric tensor-matter coupling with arbitrary charges. To cancel the isometry anomalies of the supersymmetric model based on the coset $SO(2N)/U(N)$, we use the $U(N)$ representations descending the spinor representation of $SO(2N)$. We show that only for a finite number of values of N, it is possible to construct $U(N)$ representations that satisfy the line bundle cocycle condition.

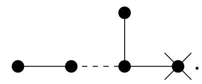
6.2 $SO(2N)$ Algebra in a $U(N)$ basis

In this section we discuss how the algebra of $SO(2N)$ can be decomposed into $SU(N)\times U(1)$ representations. We split the $SO(2N)$ generators M_{ab} into $SU(N)$ generators $T^i{}_j$, a $U(1)$ factor generator Y and broken generators $X^{ij}, \bar{X}_{ij}$ which are anti-symmetric tensors of $SU(N)$. We first discuss the embedding of $U(N)$ with generators U in $SO(2N)$, then we discuss the vector representation and finally the spinor representation.

The $\frac{2N(2N-1)}{2}$ anti-Hermitean generators M_{ab} are anti-symmetric in the indices $a, b = 1, \ldots, 2N$ of $SO(2N)$ and satisfy the commutation relations

$$[M_{ab}, M_{cd}] = \delta_{ac}M_{db} - \delta_{bd}M_{ac} - \delta_{ad}M_{cb} + \delta_{bc}M_{ad}. \tag{6.1}$$

We denote the N^2 generators of $U(N)$ by $U^i{}_j$ $(i,j=1,\ldots,N)$. The remaining $\frac{2N(2N-1)}{2}-N^2=2\frac{N(N-1)}{2}$ generators form two anti-symmetric tensor representations of $U(N)$: X^{ij} and $\bar{X}_{ij}$. Each of them has $\frac{N(N-1)}{2}$ independent components. The painted Dynkin diagram of this coset is



(6.2)

The $U(N)$ generators satisfy the algebra

$$[U^i{}_j, U^k{}_l] = \delta^i{}_l U^k{}_j - \delta^k{}_j U^i{}_l. \tag{6.3}$$

We decompose the $SO(2N)$ algebra in $U(N)$ representations by writing the $SO(2N)$ generators M_{ab} using indices $i,j=1,\ldots,N$ as

$$\begin{array}{lcl} M_{ij} & = & \frac{1}{2}(-X^{ij}-\bar{X}_{ij}-U^i{}_j+U^j{}_i), \\ M_{i\,j+N} & = & \frac{i}{2}(X^{ij}-\bar{X}_{ij}-U^i{}_j-U^j{}_i), \\ M_{i+N\,j+N} & = & \frac{1}{2}(X^{ij}+\bar{X}_{ij}-U^i{}_j+U^j{}_i). \end{array} \tag{6.4}$$

Inversely, we can express $U^i{}_j, X^{ij}$ and $\bar{X}_{ij}$ as $U^i{}_j = A^i{}_j + iS^i{}_j$ with

$$A^i{}_j = -\frac{1}{2}\left(M_{ij}+M_{i+N\,j+N}\right), \qquad S^i{}_j = \frac{1}{2}\left(M_{i\,j+N}+M_{j\,i+N}\right) \tag{6.5}$$

and $X^{ij}=-iQ^{ij}-P^{ij}$ and $\bar{X}_{ij}=iQ^{ij}-P^{ij}$ with

$$P^i{}_j = \frac{1}{2}\left(M_{i\,j}-M_{i+N\,j+N}\right), \qquad Q^i{}_j = \frac{1}{2}\left(M_{i\,j+N}-M_{j\,i+N}\right). \tag{6.6}$$

The U(1)-factor generator Y in $U(N)$ is defined as minus twice the sum of the $U(N)$ generators

$$Y = -2\sum_i^N U^i{}_i = -i2S^i{}_i = -2iM_{i\,i+N}. \tag{6.7}$$

On the other hand, the remaining $SU(N)$ generators $T^i{}_j$ are defined as the traceless part of $U^i{}_j$:

$$T^i{}_j = U^i{}_j + \frac{1}{2N} Y \delta^i{}_j. \tag{6.8}$$

Using the $U(N)$ generators $U^i{}_j$ and the broken generators X^{ij} and $\bar{X}_{ij}$ the $SO(2N)$ algebra (6.1) takes the form

$$\begin{gathered} [U^i{}_j, U^k{}_l] = \delta^i{}_l U^k{}_j - \delta^k{}_j U^i{}_l, \qquad [X^{ij}, X^{kl}] = [\bar{X}_{ij}, \bar{X}_{kl}] = 0, \\ [\bar{X}_{ij}, X^{kl}] = -\delta^k{}_i U^l{}_j - \delta^l{}_j U^k{}_i + \delta^l{}_i U^k{}_j + \delta^k{}_j U^l{}_i, \\ [U^i{}_j, \bar{X}_{kl}] = \delta^i{}_k \bar{X}_{jl} - \delta^i{}_l \bar{X}_{jk}, \quad [U^i{}_j, X^{kl}] = \delta^l{}_j X^{ik} - \delta^k{}_j X^{il}. \end{gathered} \tag{6.9}$$

This algebra satisfies the Jacobi identities. The $SO(2N)$ generators carry the following $U(1)$ charges:

$$U(1)\text{-charges of } (Y, T^i{}_j, X^{ij}, \bar{X}_{ij}) = (0, 0, 4, -4), \tag{6.10}$$

that can be obtained by computing the commutator with Y. Here we have chosen the $U(1)$ charges such that they match the convention of Slansky [44].

6.2.1 The vector representation of SO(2N)

In the vector representation of $SO(2N)$, the generators M_{ab} take the form: $(M_{ab})_{cd} = \delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad}$. Therefore an element of the $SO(2N)$-algebra reads

$$\Theta \equiv (-a^{ij}A_{ij} - s^{ij}S_{ij}) + (q^{ij}Q_{ij} - p^{ij}P_{ij}) = \begin{pmatrix} a & -s \\ s & a \end{pmatrix} + \begin{pmatrix} -p & q \\ q & p \end{pmatrix}, \tag{6.11}$$

where a, p, q are $N \times N$ real anti-symmetric matrices and s is an $N \times N$ real symmetric matrix; they are the parameters of the $SO(2N)$-algebra elements. Here we have used the definitions of the algebra elements A, S, P and Q defined in eqs. (6.5) and (6.6). The U(1)-factor generator Y (6.7) in the vector representation takes the form

$$Y = -2i \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix}. \tag{6.12}$$

In the following, it is more convenient to work in a basis in which Y is diagonal. Using a unitary transformation V we can diagonalize Y:

$$Y_D \equiv VYV^\dagger = 2 \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} \quad \text{with} \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\!\!1 & -i1\!\!1 \\ -i1\!\!1 & 1\!\!1 \end{pmatrix}. \tag{6.13}$$

We use the subscript notation A_D on any $2N \times 2N$-matrix A to indicate that A is evaluated in the basis where Y is diagonal. The effect of this similarity transformation on an element Θ of the $SO(2N)$ Lie algebra (6.11) is given by

$$\Theta_D = V\Theta V^\dagger = \begin{pmatrix} a - is & q - ip \\ q + ip & a + is \end{pmatrix} = \begin{pmatrix} u & -x^\dagger \\ x & -u^T \end{pmatrix}, \tag{6.14}$$

where $u = -u^\dagger = a - is$, $u^T = a + is$, $x = q + ip$ and $x^\dagger = -q + ip$. This coincides with the $SO(2N)/U(N)$ entry in table 4.1. Notice that in the basis where Y is diagonal, the defining property $g^{-1} = g^T$ of $SO(2N)$ becomes

$$g_D^{-1} = \mathfrak{K} g_D^T \mathfrak{K} \quad \text{with} \quad \mathfrak{K} \equiv \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix}. \tag{6.15}$$

This property can be stated as

$$\begin{pmatrix} \alpha^{\dashv} & \beta^{\dashv} \\ \gamma^{\dashv} & \delta^{\dashv} \end{pmatrix} = \begin{pmatrix} \delta^T & \beta^T \\ \gamma^T & \alpha^T \end{pmatrix}, \tag{6.16}$$

using the notations (4.42) and (4.44) for g_D and its inverse. From now on we work in the basis where the $U(1)$ charge Y is diagonal only, dropping the subscript D.

6.2.2 $SO(2N)$ spinor decomposition in anti-symmetric tensors

The components of an arbitrary spinor $\boldsymbol{\psi}$ of $SO(2N)$ can be represented using anti-symmetric tensors $\psi_p{}^{i_1\ldots i_p}$ of $SU(N)$ with p indices as

$$\boldsymbol{\psi} = \left(\psi_0, \psi_1^{i_1}, \ldots, \psi_N^{i_1\ldots i_N}\right). \tag{6.17}$$

The inner-product of two spinors $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ is given by

$$\boldsymbol{\psi}^\dagger\boldsymbol{\phi} = \sum_{p=0}^{N} \frac{1}{p!}\psi^\dagger_{p\,i_p\ldots i_1}\phi_p{}^{i_1\ldots i_p} \tag{6.18}$$

where $\psi^\dagger_{p\,i_p\ldots i_1} = (\psi_p{}^{i_1\ldots i_p})^*$. We now construct a basis for the anti-symmetric $SU(N)$-tensors and also a basis for the $SO(2N)$-spinors. The Grassmann variables Γ^i and $\bar\Gamma_i$, that satisfy the following anti-commutation relations

$$\{\Gamma^i, \bar\Gamma_j\} = \delta^i{}_j, \qquad \{\Gamma^i, \Gamma^j\} = \{\bar\Gamma_i, \bar\Gamma_j\} = 0, \tag{6.19}$$

are introduced for this purpose in ref. [128], see also [18, 20]. We construct a Hilbert space on which these Grassmann variables act. In this Hilbert space we define the vacuum state $|0\rangle$ by $\Gamma^i|0\rangle = 0$ for any i. The ket- and bra-states

$$\mathbf{e}_{p\,i_p\ldots i_1} = \bar\Gamma_{i_p}\ldots\bar\Gamma_{i_1}|0\rangle \qquad \mathbf{e}_p^{\dagger\,i_1\ldots i_p} = \langle 0|\Gamma^{i_1}\ldots\Gamma^{i_p} \tag{6.20}$$

satisfy the orthonomality relations obtained from eq. (6.19) $\mathbf{e}_p^{\dagger\,i_1\ldots i_p}\mathbf{e}_{q\,j_q\ldots j_1} = 0$ for $p \neq q$ and

$$\mathbf{e}_p^{\dagger\,i_1\ldots i_p}\mathbf{e}_{p\,j_p\ldots j_1} = \delta^{i_1}_{[j_1}\ldots\delta^{i_p}_{j_p]}, \tag{6.21}$$

where $\delta^{i_1}_{[j_1}\ldots\delta^{i_p}_{j_p]}$ denotes the complete anti-symmetrized Kronecker-delta. Therefore the states $\mathbf{e}_{p\,i_1\ldots i_p}$ form a basis of anti-symmetric rank p tensors of $SU(N)$. Using the complete anti-symmetry it is easy to show that the number of independent vectors $\mathbf{e}_p$ with length p is equal to $\binom{N}{p}$, so the total number of vectors $\{\mathbf{e}_p\}$ is equal to 2^N. The collection of these tensor $\mathbf{e}_p$ with $0 \leq p \leq N$ form a basis for $SO(2N)$-spinors. The spinors $\boldsymbol{\psi}$ and $\boldsymbol{\psi}^\dagger$ can be expanded in this basis

$$\boldsymbol{\psi} = \sum_{p=0}^{N}\frac{1}{p!}\psi_p{}^{i_1\ldots i_p}\mathbf{e}_{p\,i_p\ldots i_1} \quad \text{and} \quad \boldsymbol{\psi}^\dagger = \sum_{p=0}^{N}\frac{1}{p!}\psi^\dagger_{p\,i_p\ldots i_1}\mathbf{e}_p^{\dagger\,i_1\ldots i_p}. \tag{6.22}$$

It is straightforward to check that in this basis the inner-product of two spinors $\boldsymbol{\psi}^\dagger\boldsymbol{\phi}$ is consistent with the definition (6.18) using the orthonormality properties (6.21).

In terms of the Clifford elements Γ^i and $\bar{\Gamma}_i$, we define $2N$ gamma-matrices Γ_a with $a = 1, \ldots, 2N$ by

$$\Gamma_a = \begin{cases} i(\Gamma^i - \bar{\Gamma}_i), & a = i = 1, \ldots, N, \\ \Gamma^i + \bar{\Gamma}_i, & a = i + N = N+1, \ldots, 2N, \end{cases} \tag{6.23}$$

that form a Clifford algebra

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}. \tag{6.24}$$

This property can be used to show that the sigma-matrices

$$M_{ab} = \frac{1}{2}\Sigma_{ab} = \frac{1}{4}[\Gamma_a, \Gamma_b], \tag{6.25}$$

are the generators of the $SO(2N)$-algebra (6.1) in the spinor representation. With respect to the spinor inner-product (6.18) the gamma-matrices are Hermitean $\Gamma_a^\dagger = \Gamma_a$ and hence the sigma-matrices are anti-Hermitean $\Sigma_{ab}^\dagger = -\Sigma_{ab}$. Furthermore it implies that the Hermitean conjugation for the Grassmann variables is given by $(\Gamma^i)^\dagger = \bar{\Gamma}_i$ and $(\bar{\Gamma}_i)^\dagger = \Gamma^i$. For products of Grassmann variables A and B we have $(AB)^\dagger = B^\dagger A^\dagger$. The Hermitean chirality operator $\tilde{\Gamma}$ defined by

$$\tilde{\Gamma} = (-)^{\frac{1}{2}N(N-1)} i^{-N} \prod_{a=1}^{2N} \Gamma_a \tag{6.26}$$

can be written in terms of Clifford elements as

$$\tilde{\Gamma} = \prod_i [\Gamma^i, \bar{\Gamma}_i] = \prod_{i=1}^{N} (1 - 2\hat{n}_i) = (-)^{\hat{n}}. \tag{6.27}$$

Here we have defined the ith number operator $\hat{n}_i = \bar{\Gamma}_i \Gamma^i$ and the total number operator $\hat{n} = \sum_i \hat{n}_i$. Using this chirality operator, we can define positive and negative chirality spinors in $2N$ dimensions: $\tilde{\Gamma}\boldsymbol{\psi}_\pm = \pm\boldsymbol{\psi}_\pm$. Using the form of the chirality operator (6.27), it follows that the components of a positive chiral spinor are completely anti-symmetric tensors of even length p, while a negative chiral spinor has tensor components with odd length p.

The generators $U^i{}_j$ can be expressed in terms of the Grassmann variables as

$$U^i{}_j = -\frac{1}{2}[\Gamma^i, \bar{\Gamma}_j], \tag{6.28}$$

which satisfy the $U(N)$ algebra (6.3). Furthermore the broken $SO(2N)$ generators X^{ij} and $\bar{X}_{ij}$ can be represented by

$$X^{ij} = \Gamma^i \Gamma^j \quad \text{and} \quad \bar{X}_{ij} = \bar{\Gamma}_i \bar{\Gamma}_j.$$

The $U(1)$ charge operator (6.7) can be given in terms of the total number operator $\hat{n}$ by

$$Y = \sum_i [\Gamma^i, \bar{\Gamma}_i] = N - 2\hat{n}. \tag{6.29}$$

An anti-symmetric tensor with p indices has charge $N-2p$: $Y\mathbf{e}_p = (N-2p)\mathbf{e}_p$.

We define the dual vectors $\mathbf{e}_{\overline{N-p}}$ and $\mathbf{e}^\dagger_{\overline{N-p}}$ of the basis vectors $\mathbf{e}_p$ and $\mathbf{e}^\dagger_p$, respectively, by

$$\mathbf{e}_{\overline{N-p}}{}^{i_{p+1}\ldots i_N} = \frac{1}{p!}\mathbf{e}_{p\,i_p\ldots i_1}\epsilon^{i_1\ldots i_N} \quad \text{and} \quad \mathbf{e}^\dagger_{\overline{N-p}\,i_N\ldots i_{p+1}} = \frac{1}{p!}\epsilon_{i_N\ldots i_1}\mathbf{e}^{\dagger\, i_1\ldots i_p}_p. \tag{6.30}$$

The dual components $\psi_{\overline{p}}$ and $\bar{\psi}_{\overline{p}}$ of ψ_{N-p} and $\bar{\psi}_{N-p}$, respectively, are defined in a similar fashion. Notice that under dualization, the charge does not change, only the number of indices does.

6.2.3 Anomaly cancellation of the spinor representation

We now show that the positive chirality spinor of $SO(2N)$ has no pure $U(1)$ anomaly, unless $SO(2N)$ is isomorphic to a non-anomaly-free unitary group, $SO(2) \cong U(1)$ or $SO(6) \cong SU(4)$. The Y^k-anomaly $A_\pm(Y^k;N) = \mathrm{Tr}_\pm Y^k$ for the $\pm$ chirality spinor representation is given by

$$A_\pm(Y^k;N) = \sum_{l=0}^{N} \binom{N}{l} \frac{1 \pm (-)^l}{2}(N-2l)^k. \tag{6.31}$$

This follows using the multiplicities $\binom{N}{l}$ and charges $N-2l$ of the states $\psi_l{}^{i_1\ldots i_l}$, obtained in the previous subsection. The factor $\frac{1\pm(-)^l}{2}$ is introduced to project onto the positive or negative chirality states. The necessary details to obtain these results can be found in section 6.2.2. To calculate these anomalies we introduce the functions $q_\pm(x) = \pm\frac{1}{x} + x$ of a variable x. We define

$$P_\pm(x;N) = \frac{1}{2}\left[(q_+)^N \pm (q_-)^N\right] = \sum_{l=0}^{N} \binom{N}{l} \frac{1 \pm (-)^l}{2} x^{N-2l}. \tag{6.32}$$

Notice that the charge operator Y can be represented by $Y = x\frac{d}{dx}$, because $Yx^{N-2l} = (N-2l)x^{N-2l}$. The anomaly $A_\pm(Y^k)$ can be calculated using the functions $P_\pm$ by

$$A_\pm(Y^k;N) = \left(x\frac{d}{dx}\right)^k P_\pm(x;N)\bigg|_{x=1}. \tag{6.33}$$

To compute this we use the properties of the functions $q_\pm(x)$

$$x\frac{d}{dx}q_\pm = q_\mp, \quad q_+(1) = 2, \quad \text{and} \quad q_-(1) = 0. \tag{6.34}$$

We obtain the following results for the Y- and Y^3-anomalies in $D = 4$ dimensions

$$A_\pm(Y;N) = \begin{cases} \pm 1 & N = 1, \\ 0 & N \neq 1, \end{cases} \quad \text{and} \quad A_\pm(Y^3;N) = \begin{cases} \pm 3!2^2 & N = 3, \\ \pm 1 & N = 1, \\ 0 & N \neq 1, 3. \end{cases} \tag{6.35}$$

Hence we see that the cases $N = 1$ and $N = 3$ have indeed an anomalous spinor representation. We conclude from this anomaly analysis that for $N = 2$ and $N \geq 4$ the spinor representation of $SO(2N)$ is $U(1)$ anomaly-free.

6.3 Kähler and Killing Potentials

We now construct the Kähler potential for the coset spaces $SO(2N)/U(N)$ using the BKMU method [96, 97]. We apply this method to the $2N$ dimensional vector representation of $SO(2N)$. The BKMU projector η_- projects (4.50) on the part of this vector representation with negative Y-charge, which is an N dimensional vector representation of $SU(N)$. The coset spaces $SO(2N)/U(N)$ and $SO^{\mathbb{C}}(2N)/\hat{U}(N)$ are isomorphic because $SO^{\mathbb{C}}(2N)$, the complexification of $SO(2N)$, and $\hat{U}(N)$ is defined as the group generated by all generators of $U(N)$ together with the broken generators X^{ij} over the complex numbers, see section 4.3. The representative $\xi(z) \in SO^{\mathbb{C}}(2N)/\hat{U}(N) \cong SO(2N)/U(N)$ of the equivalence class $\xi(z)\hat{U}(N)$ is given in terms of the $\frac{1}{2}N(N-1)$ coordinates z^{ij} by

$$\xi(z) = \exp Z = \begin{pmatrix} \mathbb{1} & 0 \\ z & \mathbb{1} \end{pmatrix}, \qquad Z = -\frac{i}{2}z^{ij}\bar{X}_{ij}. \tag{6.36}$$

On the right-hand side of the equation for $\xi(z)$ we used the vector representation in the diagonal $U(1)$ charge Y basis. In the vector representation Z is nilpotent $Z^2 = 0$. The normalization factor $-\frac{i}{2}$ in the definition of Z is chosen such that we obtain the simple matrix expression for $\xi(z)$ expressed in terms of z which coincides with (4.45). Notice the distinction between z and Z: Z is the linear combination of negatively charged broken generators $\bar{X}_{ij}$ contracted with the complex coordinates of the coset space z^{ij}. Therefore Z is a $2N \times 2N$-matrix while z is an $N \times N$-matrix. Using the projection operator η_- defined in eq. (4.50) and $(\xi(z))^{-1} = \xi(-z)$ the Kähler potential is given by (4.64) and (4.60)

$$K_\sigma(\bar{z}, z) = \ln\det{}_{\eta_-}[\xi(-z)\xi^\dagger(-\bar{z})] = \ln\det\chi^{-1}, \qquad \chi^{-1} = \mathbb{1} + z\bar{z}. \tag{6.37}$$

Here $\det_{\eta_-}$ denotes that the determinant is defined on the subspace on which the projection η_- acts as the identity. Notice that the submetric $\tilde{\chi}$ defined in (4.51) is the transposed $\tilde{\chi} = \chi^T$.

We next determine the non-linear transformations of the anti-symmetric coordinates z^{ij} under a finite $g \in SO(2N)$ transformation. From eq. (4.47) we find that

$$ {}^g z = (\gamma + \delta z)(\alpha + \beta z)^{-1}, \tag{6.38} $$

using the notation introduced in eq. (4.44). The submetric χ transforms under these finite $SO(2N)$-transformations as

$$ \chi({}^g\bar{z}, {}^g z) = (\hat{h}_-^\dagger)^{-1} \chi(\bar{z}, z) (\hat{h}_-)^{-1}, \qquad \hat{h}_-(z; g) = (\delta^{\dashv} - z\beta^{\dashv})^{-1}, \tag{6.39} $$

according to eq. (4.48). Notice eq. (6.16) implies that $\hat{h}_+(z; g) = \hat{h}_-^T(z; g)$ is the transposed of $\hat{h}_-$, therefore we only use $\hat{h}_-$. The Kähler potential (6.37) transforms as

$$ K_\sigma({}^g\bar{z}, {}^g z) = K_\sigma(\bar{z}, z) + F(z; g) + \bar{F}(\bar{z}; g) \tag{6.40} $$

where the holomorphic function $F(z; g)$ is given by

$$ F(z; g) = \ln \det \hat{h}_-(z; g) \tag{6.41} $$

The complex Hermitean metric of the coset is obtained from the Kähler potential (6.37) in the standard way as the second mixed derivative

$$ g_\sigma(d\bar{z}, dz) = \mathrm{tr}\left(dz\, \chi^T\, d\bar{z}\, \chi\right) = \mathrm{tr}\left(dz \left(1\!\!1 + \bar{z}z\right)^{-1} d\bar{z} \left(1\!\!1 + z\bar{z}\right)^{-1}\right). \tag{6.42} $$

Next we discuss Killing potential M_σ for the σ-model fields z and $\bar{z}$. The Killing potentials $M_{\sigma\, u}, M_{\sigma\, x^\dagger}, M_{\sigma\, x}$ defined by eq. (2.17), can be written for the coset $SO(2N)/U(N)$ as

$$ M_\sigma(u, x, \bar{x}) = \mathrm{Tr}(\Theta \tilde{M}_\sigma) = \mathrm{tr}(u M_{\sigma\, u} + x M_{\sigma\, x^\dagger} + x^\dagger M_{\sigma\, x}). \tag{6.43} $$

The trace Tr is over $2N \times 2N$ matrices, while the trace tr is over $N \times N$ matrices. We have used similar the notation as in eq. (6.14)

$$ \Theta = \begin{pmatrix} u & -x^\dagger \\ x & -u^T \end{pmatrix} \quad \text{and} \quad \tilde{M}_\sigma = \begin{pmatrix} \tilde{M}_{\sigma\, u} & \tilde{M}_{\sigma\, x^\dagger} \\ -\tilde{M}_{\sigma\, x} & -\tilde{M}_{\sigma\, u^T} \end{pmatrix}. \tag{6.44} $$

Hence we find that $M_{\sigma\, x} = \tilde{M}_{\sigma\, x}$, $M_{\sigma\, x^\dagger} = \tilde{M}_{\sigma\, x^\dagger}$ and $M_{\sigma\, u} = \tilde{M}_{\sigma\, u} + \left(\tilde{M}_{\sigma\, u^T}\right)^T$. We now determine the Killing potentials explicitly. We introduce some notation that might seem some what cumbersome at this stage, but which will be convenient when we discuss the Killing potentials due to matter coupling. Define the matrices R and R_T by

$$ R(z; \Theta) = x - u^T z - zu + zx^\dagger z, \qquad R_T(z; \Theta) = -u^T + zx^\dagger. \tag{6.45} $$

Notice that $\delta z = R(z;\Theta)$ is a compact notation for the Killing vectors of the coset space. Computing the Killing potentials M_σ in the standard way gives

$$-iM_\sigma(\bar{z}, z; \Theta) = -\mathrm{tr}\,\Delta(\bar{z}, z; \Theta), \tag{6.46}$$

where we have defined the matrix Δ, in analogy to the Killing potentials associated with the Grassmannian cosets (5.25), by

$$\Delta(\bar{z}, z; \Theta) \equiv R_T - R\bar{z}\chi = (zu\bar{z} - u^T - x\bar{z} + zx^\dagger)\chi. \tag{6.47}$$

The matrix Δ can also be written in terms of the BKMU variable $\xi(z)$ and Θ and the projector $\tilde{\eta}_-^T = \begin{pmatrix} 0 & \mathbb{1} \end{pmatrix}$ as

$$\Delta(\bar{z}, z; \Theta) = \tilde{\eta}_-^T \left(\xi(z)\right)^{-1} \Theta \left(\xi^\dagger(\bar{z})\right)^{-1} \tilde{\eta}_- \,\chi. \tag{6.48}$$

The Killing potential matrix $\tilde{M}_\sigma$ is given by

$$-i\tilde{M}_\sigma = -\xi^\dagger(-\bar{z})\tilde{\eta}_- \,\chi\, \tilde{\eta}_-^T \xi(-z) = -\begin{pmatrix} \bar{z}\chi z & -\bar{z}\chi \\ -\chi z & \chi \end{pmatrix}. \tag{6.49}$$

From this we read off the Killing potentials

$$-iM_{\sigma\, x} = -\chi z, \quad -iM_{\sigma\, x^\dagger} = \bar{z}\chi, \quad -iM_{\sigma\, u} = -2\bar{z}\chi z + \mathbb{1}. \tag{6.50}$$

6.3.1 Matter coupling

We now discuss the different types of matter couplings to a supersymmetric $SO(2N)/U(N)$ model. We only need the decomposition of the chiral spinor representation of $SO(2N)$ in completely anti-symmetric $SU(N)$ tensors in our construction of anomaly-free models later, we focus here primarily on such representations.

We introduce an anti-symmetric tensor x of rank-2, that transforms in the same way as a differential dz under a finite transformation (6.38):

$${}^g x = \hat{h}_-(z; g) x \hat{h}_-^T(z; g), \tag{6.51}$$

using that $\hat{h}_+ = \hat{h}_-^T$ holds for $SO(2N)$. An invariant Kähler potential for x is given by $K(\bar{z}, \bar{x}; z, x) = \mathrm{tr}\left(x\chi^T\bar{x}\chi\right)$. Below we discuss non-linear $SO(2N)$ realizations on the irreducible completely anti-symmetric $SU(N)$-tensor representations with p indices and arbitrary rescaling charge q. We denote these tensors by $T_{(p;q)}^{i_1\ldots i_p}$, or without indices by $T_{(p;q)}$ when there is no confusion possible. We interpret them as matter multiplets and construct their invariant Kähler potentials. To define their transformation properties we first consider a vector $T^i = T^i_{(1;0)}$ without a rescaling charge. It transforms as

$${}^g T = \hat{h}_-(z; g) T, \tag{6.52}$$

under a finite non-linear $SO(2N)$ transformation (6.38). An invariant Kähler potential for the vector $T = T_{(1;0)}$ is given by

$$K_{(1;0)} = \bar{T}\chi T = \bar{T}_i\, \chi^i{}_j T^j, \tag{6.53}$$

with the metric χ defined in eq. (6.37). It is also possible to couple a singlet chiral multiplet S to the coset, which can be interpreted as a section of the minimal line bundle. It transforms as (4.81)

$$^gS = e^{\frac{1}{2}F(z)}S = (\det \hat{h}_-)^{\frac{1}{2}}S. \tag{6.54}$$

Its Kähler potential $K_{(0;1)} = S\bar{S}e^{-\frac{1}{2}K_\sigma}$ is invariant. We can rescale any given chiral multiplet with this singlet S. For example, $T^i_{(1;q)} \equiv S^q T^i_{(1;0)}$ transforms as

$$^gT_{(1;q)} = {}^g(S^q T_{(1;0)}) = e^{\frac{q}{2}F(z)}\hat{h}_- T_{(1;q)} = (\det \hat{h}_-)^{\frac{q}{2}}\hat{h}_- T_{(1;q)}. \tag{6.55}$$

Since S is a section of the minimal line bundle over the coset $SO(2N)/U(N)$ the rescaling charge q is integer. The generalization of the Kähler potential (6.53) is given by

$$K_{(1;q)} = \bar{T}_{(1;q)} g_{(1;q)} T_{(1;q)}, \quad \chi_{(1;q)} = e^{-\frac{q}{2}K_\sigma}\chi = (\det \chi)^{\frac{q}{2}}\chi. \tag{6.56}$$

By taking the completely anti-symmetric tensor products of a set of $SU(N)$ vectors $\{T_1^{i_1}, \ldots, T_p^{i_p}\}$ we obtain an $SU(N)$ tensor of rank-p with rescaling charge q

$$T^A_{(p;q)} = T^{i_1\ldots i_p}_{(p;q)} \equiv \frac{1}{p!}S^q T_1^{[i_1} * \ldots * T_p^{i_p]}. \tag{6.57}$$

Here we have introduced the multi-index notation $A = (i_1 \ldots i_p)$ and $[\ldots]$ denotes the complete anti-symmetrization of the indices inside the brackets. In analogy to the transformations of $T_{(1;0)}$ and S we obtain

$$^gT^{i_1\ldots i_p}_{(p;q)} = (\det \hat{h}_-)^{\frac{q}{2}}(\hat{h}_-)^{i_1}{}_{j_1}\ldots(\hat{h}_-)^{i_p}{}_{j_p}T^{j_1\ldots j_p}_{(p;q)}. \tag{6.58}$$

The invariant Kähler potential for this tensor $T_{(p;q)}$ is the direct generalization [69] of the Kähler potentials for the vector (6.53) and singlet (6.3.1)

$$K_{(p;q)} = \bar{T}_{(p;q)B}\, g^{B}_{(p;q)\,A}\, T^A_{(p;q)}, \quad g^{B}_{(p;q)\,A} = \frac{1}{p!}(\det \chi)^{\frac{q}{2}}\chi^{j_1}{}_{i_1}\ldots\chi^{j_p}{}_{i_p}. \tag{6.59}$$

The Levi-Cevita tensor $\epsilon_{i_1\ldots i_N}$ is invariant under $SU(N)$ transformations. We use it to define a dual-tensor $T_{(\overline{N-p};q)\,i_{p+1}\ldots i_N}$ with $N-p$ indices and rescaling charge q by

$$T_{(\overline{N-p};q)\,i_{p+1}\ldots i_N} \equiv \frac{1}{p!}T^{i_p\ldots i_1}_{(p;q)}\epsilon_{i_1\ldots i_N}, \tag{6.60}$$

which transforms under the finite transformation (6.38) as

$$ {}^gT_{(\overline{p};q)i_1\dots i_p} = T_{(\overline{p};q)j_1\dots j_p}(\hat{h}_-^{-1})^{j_1}{}_{i_1}\dots(\hat{h}_-^{-1})^{j_p}{}_{i_p}(\det\hat{h}_-)^{1+\frac{q}{2}}. \tag{6.61} $$

The power $1+\frac{q}{2}$ of $\det\hat{h}_-$ instead of $\frac{q}{2}$ arises because we have changed from $\hat{h}_-$ to its inverse at the expense of an additional factor of the determinant of $\hat{h}_-$. In our conventions tensors have superscript indices while dual tensors have subscript indices. As is clear from the definitions here, working with tensors or dual tensors is equivalent. The invariant Kähler potential for a dual tensor is given by

$$ K_{(\bar{p};q)} = T_{(\overline{p};q)\,A}\,g^{A}_{(\overline{p};q)\,B}\,\bar{T}^{B}_{(\overline{p};q)}, \quad g^{A}_{(\overline{p};q)\,B} = \frac{1}{p!}(\det\chi)^{1+\frac{q}{2}}(\chi^{-1})^{i_1}{}_{j_1}\dots(\chi^{-1})^{i_p}{}_{j_p}. \tag{6.62} $$

We end this subsection with a discussion of the Killing potentials $M_{(p;q)}$ and $M_{(\bar{p};q)}$ for a tensor $T_{(p;q)}$ and a dual-tensor $\bar{T}_{(\bar{p};q)}$ of rank-p with a rescaling charge q, respectively. As the Kähler potentials $K_{(p;q)}$ and $K_{(\bar{p};q)}$ are invariant the Killing potentials are obtained by

$$ -iM_{(p;q)} = K_{(p;q),\mathcal{A}}\mathcal{R}^{\mathcal{A}}, \qquad -iM_{(\bar{p};q)} = K_{(\bar{p};q),\mathcal{A}}\mathcal{R}^{\mathcal{A}}, \tag{6.63} $$

where $\delta_i Z^{\mathcal{A}} = \mathcal{R}_i^{\mathcal{A}}$ denote the Killing vectors

$$ \begin{aligned} \delta z &= R, \\ \delta T^{i_1\dots i_p}_{(p;q)} &= \sum_{r=1}^{p}(R_T)^{i_r}{}_j T^{i_1\dots j\dots i_p}_{(p;q)} + \frac{q}{2}\,\mathrm{tr}(R_T)\,T^{i_1\dots i_p}_{(p;q)}, \\ \delta T_{(\bar{p};q)i_1\dots i_p} &= \sum_{r=1}^{p} T_{(\bar{p};q)i_1\dots j\dots i_p}(-R_T)^j{}_{j_r} + \Big(1+\frac{q}{2}\Big)\mathrm{tr}(R_T)\,T_{(\bar{p};q)\,i_1\dots i_p}. \end{aligned} \tag{6.64} $$

They follow from expanding the finite transformations (6.38), (6.58) and (6.61) to first order in the infinitesimal parameters $u, x, x^\dagger$. The Killing potential for a rank-p tensor with rescaling charge q is given by

$$ -iM_{(p;q)} = \bar{T}_{(p;q)B}\,g^{B}_{(p;q)C}\Delta^{C}_{(p;q)A}T^{A}_{(p;q)}, \tag{6.65} $$

where we have defined in terms of eq. (6.47):

$$ \Delta^{C}_{(p;q)A} = \sum_{r=1}^{p}\delta^{k_1}{}_{i_1}\dots\Delta^{k_r}{}_{i_r}\dots\delta^{k_p}{}_{i_p} + \frac{q}{2}\mathrm{tr}\Delta\,\delta^{k_1}{}_{i_1}\dots\delta^{k_p}{}_{i_p}. \tag{6.66} $$

To obtain this result we have made the following steps. First, we obtained the Killing potential for a rank-1 tensor (a vector) with rescaling charge zero. This result can easily be generalized to a rank-p tensor with rescaling charge zero. Next we construct the Killing potential for a rank-0 tensor (a singlet) with an arbitrary rescaling charge. Finally we put all results together to obtain eq. (6.65).

We can proceed in a similar fashion to obtain the Killing potential $M_{(\bar{p};q)}$ for a rank-p dual-tensor with a rescaling charge q. As the dualization has introduced a determinant $\det \hat{h}_-$ in the finite transformation (6.61), it is more convenient to first consider a rank-p dual tensor with rescaling charge -2, which precisely cancels the determinant. To obtain the final result for a rank-p dual-tensor with a rescaling charge q, we have to rescale again, which introduces a factor $1 + \frac{q}{2}$. Finally, the Killing potential reads

$$-iM_{(\bar{p};q)} = T_{(\bar{p};q)B} \Delta^{B}_{(\bar{p};q)C} g^{C}_{(\bar{p};q)A} \bar{T}^{A}_{(\bar{p};q)}, \tag{6.67}$$

with $\Delta^{C}_{(\bar{p};q)A}$ defined by

$$\Delta^{B}_{(\bar{p};q)C} = \sum_{r=1}^{p} \delta^{j_1}{}_{k_1} \dots (-\Delta)^{j_r}{}_{k_r} \dots \delta^{j_p}{}_{k_p} + \left(1 + \frac{q}{2}\right) \mathrm{tr}\, \Delta\, \delta^{j_1}{}_{k_1} \dots \delta^{j_p}{}_{k_p}. \tag{6.68}$$

6.4 Consistent $SO(2N)/U(N)$ spinor model

In this section we construct an anomaly-free model based on the spinor representation of $SO(2N)$ that contains the coordinates of the coset $SO(2N)/U(N)$. There are only a finite number of these models that satisfies the line bundle constraint.

A supersymmetric model based on the $SO(2N)/U(N)$ coset space is not free of anomalies by itself as all the $\frac{1}{2}N(N-1)$ anti-symmetric coordinates z^{ij} and therefore also their chiral fermionic partners carry the same charge, 4 in the standard normalization. To construct a consistent supersymmetric model around this coset, we try to embed the coordinates in an anomaly-free representation. All representations of $SO(2N)$ are anomaly-free, unless $SO(2N)$ is isomorphic to a non-anomaly-free unitary group. This happens for $SO(2) \cong U(1)$ and $SO(6) \cong SU(4)$, hence we disregard the cases $N = 1, 3$ below. In the section 6.2.3 we derived these results by calculating the possible $U(1)$ anomalies of the chiral spinor representation. The $SO(2N)$ spinor representation branches to an anti-symmetric 2-tensor of $SU(N)$. The other $SU(N)$ tensors, which descend from the spinor representation, transform under the full $SO(2N)$ transformations. Global consistency requires that these matter representations are sections of bundles. If one of these sections is a line bundle we run into the cocycle condition, which greatly restricts the freedom of charge assignments. In section 4.4 we have determined a characterization of a section of the minimal line bundle over $SO(2N)/U(N)$. As the dimension $2N$ is even, the irreducible representations carry definite chirality. We show that it is sufficient to consider only the positive chiral spinor representation for our purpose of extending the coset to an anomaly-free model. After that we turn to the main result of this section: the cocycle condition only allows for a very restricted class of consistent $SO(2N)/U(N)$ spinor models: $N = 2, 5, 6, 8$.

For the construction of a consistent model of $SO(2N)/U(N)$ based on irreducible spinor representation, we need to identify the anti-symmetric coordinates z^{ij} of the coset space with an anti-symmetric 2-tensor of the branching of the spinor. The tensors ${\psi_2}^{ij}$ or $\psi_{\bar{2}\,ij}$ are possible candidates. The charge of ${\psi_2}^{ij}$ is $N-4$ and it has positive chirality. The charge of $\psi_{\bar{2}\,ij}$ is opposite and its chirality is $(-)^N$. Notice that for $N=4$ we can never construct a consistent model using the spinor representation as the charges of ${\psi_2}^{ij}$ and $\psi_{\bar{2}\,ij}$ are zero, while the charge of the coordinate z^{ij} is non-zero. Therefore we consider the cases $N=2$ and $N \geq 5$ from now on, since $N=1,3$ do not lead to anomaly-free models. For N even ${\psi_2}^{ij}$ and $\psi_{\bar{2}\,ij}$ both reside in the positive chirality spinor, so that it does not matter which of the two is identified with the coordinates. On the other hand for odd N, ${\psi_2}^{ij}$ and $\psi_{\bar{2}\,ij}$ are contained in the positive and negative spinor representation, respectively. But as the relative charge of the anti-symmetric tensors of the positive and the negative spinor the same, both give the same result. We concluded that it is sufficient to consider only the positive chirality spinor representation and only the state ${\psi_2}^{ij}$ as candidate for the coordinates z^{ij} of $SO(2N)/U(N)$.

We discuss the restriction that the consistency of the line bundle poses on the construction of anomaly-free extension of cosets $SO(2N)/U(N)$ using the positive chirality spinors. It was shown in section 4.4 that the minimal charge of the line bundle over the coset space $SO(2N)/U(N)$ is equal to N when the charge of the coordinates is taken to be 4. As all tensors descending from a positive chirality spinor have an even number of indices, they can be obtained from completely anti-symmetric tensor products of the tangent vectors of the coset $SO(2N)/U(N)$ times an integer power of the minimal non-trivial singlet. In particular, the tensor $\psi_{(2p;q)}$ with length $2p$ and with rescaling charge $q(p;N)$ has a Y-charge $4p+Nq(p;N)$. For each p this charge should be proportional to the charge $N-4p$ of the anti-symmetric tensor with $2p$ indices within the positive chirality spinor representation. This leads to the relation $\lambda(N-4p) = 4p + Nq(p;N)$ where the proportionality factor $\lambda \in \mathbb{R}$ has to be determined. Since the components of the anti-symmetric tensor with 2 indices ($p=1$) are identified with the coordinates z^{ij} of the coset, they do not have a rescaling charge, hence we find that $\lambda = \frac{4}{N-4}$. Solving for $q(p;N)$ gives

$$q(p;N) = \frac{4}{N-4}(1-p). \tag{6.69}$$

For consistency of the line bundle we need that $q(p;N)$ is an integer for all integer $0 \leq p \leq [N/2]$. Since $q(p;N)$ is integer whenever $q(0;N)$ is integer, we only have to determine for which N this is the case. Of course, $q(0;N)$ is only an integer if $N-4$ is a divider of 4. This implies that $N = 0,2,3,5,6,8$. Clearly, $N=0$ is impossible. Though the case $N=3$ satisfies the line bundle quantization condition, it is excluded, since it does not lead to an anomaly-free model. Therefore the possible choices are:

N	2	5	6	8
$q(0;N) = \frac{4}{N-4}$	-2	4	2	1

The case of $N = 2$ is trivial in the sense that the coset is isomorphic to the simplest coset $SU(2)/U(1)$, the sphere, because $SO(4) \cong SU(2) \times SU(2)$. Notice that except for the last case $N = 8$ we only use squares of the minimal line bundle.

We finish this section by giving the Kähler potentials for the anomaly-free $SO(2N)/U(N)$ models based on the positive chiral spinor representation. The matter content is fixed by the discussion above: we need for each $0 \leq p \leq [N/2]$ a rank $2p$ completely anti-symmetric $SU(N)$ tensor $T_{(\,2p;\; q(p;N)\,)}$ with rescaling charge $q(p;N)$ given in eq. (6.69), except for $p = 1$; the anti-symmetric tensor with two indices we identify with the coordinates z^{ij} of the $SO(2N)/U(N)$ coset itself. Using the Kähler potentials for the coset (6.37) and for anti-symmetric tensor representations with an arbitrary rescaling charge (6.59), we can express the Kähler potential for the complete system by

$$\mathcal{K} = \frac{1}{2} K_\sigma + \sum_{p=0,\; p\neq 1}^{[N/2]} K_{(\,2p;\; q(p;N)\,)}. \tag{6.70}$$

Here we have included a factor $\frac{1}{2}$ to obtain the standard normalization of the kinetic terms of the Goldstone boson fields. In section 6.5 we discuss the consistent $SO(10)/U(5)$-spinor model in more detail. There we give the explicit expression for the Kähler potential, using dual tensors to reduce the number of indices.

6.5 $SO(10)/U(5)$-spinor model phenomenology

The previous sections contain all the necessary ingredients for discussing the phenomenology of the anomaly-free $SO(10)/U(5)$-spinor model. The fermionic field content of this model corresponds to one family of quarks and leptons, including a right-handed neutrino. This can be seen by looking at the $SU(5)$ representations of the chiral multiplets that the model contains: the coordinates z^{ij} form the $\underline{10}$ of $SU(5)$. The completely anti-symmetric tensor with 4 indices is equivalent to the $\underline{\bar{5}}$, which we denote by k_i, and in addition we have a singlet h of $SU(5)$. We analyze some phenomenological aspects in the context of global supersymmetry here. We explain why there is no invariant superpotential and we discuss possible gaugings and the resulting scalar potential.

We denote all chiral superfields by $\Sigma^{\mathcal{A}} = (Z^{\mathcal{A}}, \psi_L^{\mathcal{A}})$, with scalar components

$Z^{\mathcal{A}} = (z^{ij}, h, k_i)$. The Kähler potential is given by

$$\begin{aligned}\mathcal{K}(\bar{Z}, Z) &= \frac{1}{2}K_\sigma + K_{(0;4)} + K_{(\bar{1};-4)} \\ &= \frac{1}{2f^2}\ln\det\chi^{-1} + (\det\chi)^2|h|^2 + (\det\chi)^{-1}\,k\chi^{-1}\bar{k}.\end{aligned} \tag{6.71}$$

with the submetric $\chi^{-1} = \mathbb{1} + f^2 z\bar{z}$ and $e^{f^2 K_\sigma} = (\det\chi)^{-1}$. This is the explicit form of eq. (6.70) in the $SO(10)/U(5)$ case, with the inclusion of the mass parameter $f^{-1} = M_\sigma$ that sets the scale of the σ-model.

We cannot obtain an invariant superpotential. The transformations of the coordinates z make it impossible to include them in the superpotential. The singlet h and the dual vector k_i could in principle be included in the superpotential, but there is no non-vanishing holomorphic $SU(5)$ invariant for k_i. As h transforms under the $U(1)$ and there is no field that compensates for its transformation, it also cannot appear in the superpotential. This implies that the scalar potential only contains D-term contributions from gauging of (part of) the symmetries.

We now analyze the D-term scalar potential for various gaugings. The isometries of this model are generated by the Killing vectors

$$\delta_\Theta z = R, \qquad \delta_\Theta h = 2\mathrm{tr} R_T h, \qquad \delta_\Theta k = -k(R_T + \mathrm{tr}(R_T)\mathbb{1}), \tag{6.72}$$

with $R(z;\Theta) = \frac{1}{f}\,x - u^T z - zu + f\,zx^\dagger z$ and $R_T(z;\Theta) = -u^T + f\,zx^\dagger$. They can be obtained from the full Killing potential $\mathcal{M} = M_\sigma + M_{(0;4)} + M_{(\bar{1};-4)}$, which is given by

$$-i\mathcal{M} = \mathrm{tr}\Delta(-\frac{1}{2f^2} - K_{(\bar{1};-4)} + 2K_{(0;4)}) - e^{f^2 K_\sigma} k\Delta\chi^{-1}\bar{k}. \tag{6.73}$$

By substituting the expression (6.47) for Δ we obtain in index-free notation

$$\begin{aligned}-i\mathcal{M}_u &= (\mathbb{1} - 2f^2\bar{z}\chi z)\Big(-\frac{1}{2f^2} - K_{(\bar{1};-4)} + 2K_{(0;4)}\Big) - e^{f^2 K_\sigma}(\bar{z}\bar{k}kz - f^2 k^T\bar{k}^T), \\ -i\mathcal{M}_{x^\dagger} &= -f\bar{z}\chi\Big(-\frac{1}{2f^2} - K_{(\bar{1};-4)} + 2K_{(0;4)}\Big) + f e^{f^2 K_\sigma}\bar{z}\bar{k}k, \\ -i\mathcal{M}_x &= f\chi z\Big(-\frac{1}{2f^2} - K_{(\bar{1};-4)} + 2K_{(0;4)}\Big) - f e^{f^2 K_\sigma}\bar{k}kz.\end{aligned} \tag{6.74}$$

When we gauge the full $SO(10)$, the unitary gauge can be used to put all Goldstone bosons z to zero. This implies that the broken Killing potentials $\mathcal{M}_x$ and $\mathcal{M}_{x^\dagger}$ vanish automatically, leaving us with the $U(5)$ Killing potentials only. When we only gauge $U(5)$ then the Killing potentials $\mathcal{M}_x$ and $\mathcal{M}_{x^\dagger}$ are irrelevant, so again we have to consider the $U(5)$ Killing potentials. In this case it is not automatic that one can choose the vacuum expectation value $\langle z\rangle = 0$ in the minimum of the potential. To analyze both cases at once we consider the D-term potential due to gauging of $SU(5)\times U(1)$ including a Fayet-Iliopoulos term with parameter ξ for the $U(1)$

$$V = \frac{g_1^2}{2N}(\xi - i\mathcal{M}_Y)^2 + \frac{g_5^2}{2}\mathrm{tr}(-i\mathcal{M}_t)^2. \tag{6.75}$$

Here g_1 and g_5 denote the $U(1)$ and $SU(5)$ gauge couplings, respectively. In the case of the full $SO(10)$ the coupling constants are equal: $g_1 = g_5 = g_{10}$ and $\xi = 0$. We leave the rank $N = 5$ of $SO(10)$ in, to keep track of the dependence of the rank. Finally the $SU(5)$ Killing potentials $\mathcal{M}_t$ are the traceless part of $\mathcal{M}_u$ and the $U(1)$ Killing potential $\mathcal{M}_Y$ is the trace of $\mathcal{M}_u$:

$$\mathcal{M}_u = \mathcal{M}_t + \frac{1}{N}\mathcal{M}_Y\,\mathbb{1}, \qquad \mathcal{M}_Y = \mathrm{tr}\mathcal{M}_u. \tag{6.76}$$

We can derive $\mathrm{tr}\mathcal{M}_t^2$ from $\mathcal{M}_Y$ and $\mathrm{tr}\mathcal{M}_u^2$ by

$$\mathrm{tr}(-i\mathcal{M}_t)^2 = \mathrm{tr}(-i\mathcal{M}_u)^2 - \frac{1}{N}(-i\mathcal{M}_Y)^2. \tag{6.77}$$

When one substitutes this into the scalar potential (6.75) in the case where $g_1 = g_5$ and $\xi = 0$ we find the standard $U(5)$ D-terms scalar potential, showing the relative normalization between the coupling constants is correct. We now introduce the following short-hand notations

$$A_i \equiv \mathrm{tr}(\chi^i), \quad B_i \equiv e^{f^2K_\sigma}k\chi^{-i}\bar{k} \quad \text{and} \quad D = -\frac{1}{2f^2} - K_{(\bar{1};-4)} + 2K_{(0;4)} \tag{6.78}$$

to obtain compact expressions for the (squares) of the Killing potentials. In the unitary gauge $z = \bar{z} = 0$, we have $A_i = N$ and $B_i = B_0 = k\bar{k}$ for all i. After some straightforward algebra one finds

$$\begin{aligned} -i\mathcal{M}_Y =& (-N + 2A_1)D + 2B_0 - B_1, \\ \mathrm{tr}(-i\mathcal{M}_u)^2 =& (N - 4A_1 + 4A_2)D^2 - 2(-B_1 + 4B_0 - 4B_{-1})D + (B_1 - B_0)^2 + B_0^2. \end{aligned} \tag{6.79}$$

We first consider the situation in the unitary gauge; then the Killing potentials simplify considerably and the scalar potential becomes

$$\begin{aligned} V_{uni} =& \frac{g_1^2}{2N}(\xi + ND + B_0)^2 + \frac{g_5^2}{2}\frac{n-1}{n}B_0^2 \\ =& \frac{g_1^2}{2N}\Big(\xi - \frac{N}{2f^2} + 2N|h|^2 - (N+1)|k|^2\Big)^2 + \frac{g_5^2}{2}\frac{N-1}{N}\Big(|k|^2\Big)^2. \end{aligned} \tag{6.80}$$

From this we see that we only have a supersymmetric minimum if

$$|k|^2 = 0, \qquad |h|^2 = \frac{1}{4f^2} - \frac{1}{2N}\xi. \tag{6.81}$$

Therefore if the Fayet-Iliopoulos parameter $\xi > \frac{N}{2f^2}$, the unitary gauge (putting $z = \bar{z} = 0$) no longer corresponds to a supersymmetric minimum of the potential. For $\xi < \frac{N}{2f^2}$ we see that $z = \bar{z} = 0$ is compatible with a supersymmetric minimum. This conclusion holds, in particular, if we gauge the full $SO(10)$ as $\xi = 0$ in that case. Since h acquires a non-vanishing vacuum expectation value, the $U(1)$ is broken.

We now determine all supersymmetric vacua at the classical level. Using the Killing potentials $\mathcal{M}_Y$ and $\mathcal{M}_u$ supersymmetric vacua are characterized by the equations

$$0 = \xi - i\mathcal{M}_Y, \qquad \mathrm{tr}(-i\mathcal{M}_u)^2 = \frac{1}{N}(-i\mathcal{M}_Y)^2 = \frac{1}{N}\xi^2. \tag{6.82}$$

To discuss the solutions of these equations, it is convenient to introduce some additional notation. Denote the length of the dual vector k by $C = k\bar{k}$ and let $\hat{k}$ be the unit vector $\hat{k}\bar{\hat{k}} = 1$ in the direction of k. And finally define $\hat{B}_i = e^{f^2 K_\sigma}\hat{k}\chi^{-i}\bar{\hat{k}}$. We take the vacuum expectation values of z and $\hat{k}$ arbitrary and determine the values of C and D. Notice that not all values of C and D are allowed because they are functions of the absolute value squared of h and k. In fact we find two inequalities

$$D + \hat{B}_1 C + \frac{1}{2f^2} \geq 0, \qquad C \geq 0. \tag{6.83}$$

The supersymmetric vacuum equations (6.82) are written as

$$V^T H V = \frac{1}{N}\xi^2, \qquad W^T V = \xi \tag{6.84}$$

where H is a symmetric 2×2-matrix and V, W a vector

$$V = \begin{pmatrix} D \\ C \end{pmatrix}, \qquad H = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad W = \begin{pmatrix} 2A_1 - N \\ \hat{B}_1 - 2\hat{B}_0 \end{pmatrix} \tag{6.85}$$

with $a = N - 4A_1 + 4A_2 = \mathrm{tr}(1\!\!1 - 2\chi)^2$, $b = \hat{B}_1 - 4\hat{B}_0 + 4\hat{B}_{-1}$, $c = (\hat{B}_1 - \hat{B}_0)^2 + \hat{B}_0^2$, Observe that a and c are both non-negative. The matrix H is real and symmetric, hence it has real eigenvalues

$$\lambda_\pm = \frac{a + c \pm \sqrt{(2b)^2 + (a-c)^2}}{2}. \tag{6.86}$$

Observe that the eigenvalue λ_+ is always positive. For λ_- on the other hand we find that: $\lambda_- < 0$ when $b^2 > ac$, $\lambda_- = 0$ when $b^2 = ac$ and $\lambda_- > 0$ when $b^2 < ac$. The eigenvectors $e_\pm$ of H can be given by

$$e_\pm = \frac{1}{n_\pm}\begin{pmatrix} -b \\ a - \lambda_\pm \end{pmatrix}, \qquad n_\pm^2 = b^2 + (a - \lambda_\pm)^2 \tag{6.87}$$

and they satisfy

$$He_{\pm} = \lambda_{\pm} e_{\pm}, \qquad e_{\alpha}^{T} e_{\beta} = \delta_{\alpha\beta} \quad \text{with} \quad \alpha, \beta = \pm. \tag{6.88}$$

Using this orthonormal basis where the vectors V and W have components $V^{\pm}$ and $W^{\pm}$,

$$V = V^{\alpha} e_{\alpha}, \quad \text{and} \quad W = W^{\alpha} e_{\alpha}, \tag{6.89}$$

the supersymmetric vacua equations take the form

$$\lambda_{+}(V^{+})^2 + \lambda_{-}(V^{-})^2 = \frac{1}{N}\xi^2, \qquad W^{+}V^{+} + W^{-}V^{-} = \xi. \tag{6.90}$$

Since $\lambda_{+} > 0$ is always positive, we can solve the inhomogeneous quadratic equation for V^{+}. Combining it with the linear equation, we find

$$\left((W^{-})^2 + \frac{\lambda_{-}}{\lambda_{+}}(W^{+})^2\right)(V^{-})^2 - 2\xi W^{-}V^{-} + \xi^2\left(1 - \frac{(W^{+})^2}{N\lambda_{+}}\right) = 0. \tag{6.91}$$

If this equation has a solution, we can find a supersymmetric minimum. Let us discuss various solutions of this equation. We first analyze the situation when there is no quadratic term in this equation: $(W^{-})^2 + \frac{\lambda_{-}}{\lambda_{+}}(W^{+})^2 = 0$. This can only happen in the following cases: $(\lambda_{-} = 0, W^{-} = 0)$, $(\lambda_{-} < 0, (W^{-})^2 = \frac{|\lambda_{-}|}{\lambda_{+}}(W^{+})^2)$ and $(W^{-} = W^{+} = 0)$. We can now discriminate between three cases: $\xi = 0$, $\xi \neq 0, W^{-} = 0$ and $\xi \neq 0, W^{-} \neq 0$. In both the first and second case we find that V^{-} is undetermined. In the second case one should realize that $\lambda_{-} = 0$ for consistency. In the third case we find the solution

$$V^{-} = \frac{\xi}{2W^{-}}\left(1 - \frac{(W^{+})^2}{N\lambda_{+}}\right). \tag{6.92}$$

We now turn to the other situation $(W^{-})^2 + \frac{\lambda_{-}}{\lambda_{+}}(W^{+})^2 \neq 0$. As V^{-} is a real quantity, only real solutions of the quadratic equation (6.91) can be accepted. By computing the discriminant, we find the condition

$$0 \leq \frac{4\xi^2}{N\lambda_{+}^2} W_{+}^2 \left(\lambda_{-}(W^{+})^2 + \lambda_{+}(W^{-})^2 - N\lambda_{+}\lambda_{-}\right). \tag{6.93}$$

Therefore we have again three different situations that can lead to a supersymmetric minimum. If $\xi = 0$ then $V^{-} = 0$. If $\xi \neq 0$ and $W^{+} = 0$ then $V^{-} = \frac{\xi}{W^{-}}$. The third case has the solutions

$$V^{-} = \frac{\lambda_{+}W^{-}\xi \pm \frac{1}{\sqrt{N}}\xi W^{+}\sqrt{\lambda_{-}(W^{+})^2 + \lambda_{+}(W^{-})^2 - N\lambda_{+}\lambda_{-}}}{\lambda_{-}(W^{+})^2 + \lambda_{+}(W^{-})^2}. \tag{6.94}$$

However if $\lambda_- \geq 0$ this has only real solutions if $\lambda_-(W^+)^2 + \lambda_+(W^-)^2 \geq N\lambda_+\lambda_-$ and if $\lambda_- < 0$ then $-N\lambda_+\lambda_- + \lambda_+^2(W^-)^2 \geq \lambda_-(W^+)^2$. Notice that if not one of these equations is satisfied, then there is no solution hence no supersymmetric minimum is possible. For all the solutions, we have to check whether they are allowed by eqs. (6.83). By combining these inequalities with the second relation of eq. (6.90) we obtain the following two restrictions

$$\left[\frac{a-\lambda_-}{n_-}W^+ - \frac{a-\lambda_+}{n_+}W^-\right]V^- \geq \frac{\lambda_+ - a}{n_+}\xi,$$

$$\begin{aligned}\left[\left(\hat{B}_1(\lambda_- - a) + b\right)\frac{W^+}{n_-} - \left(\hat{B}_1(a-\lambda_+) - b\right)\frac{W^-}{n_+}\right]V^- \\ \geq \left(\hat{B}_1(a-\lambda_+) - b\right)\frac{\xi}{n_+} - \frac{W^+}{2f^2}.\end{aligned} \tag{6.95}$$

Because all the parameters $W^\pm, a, b, c$ are arbitrary and independent, it is hard to make statements without a numerical analysis. Such an analysis of the vacuum manifold is outside the scope of this thesis and requires further development.

Chapter 7

Supersymmetric $E_6/[SO(10) \times U(1)]$ Model

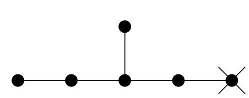
7.1 Introduction

This chapter is devoted to the coset space $E_6/[SO(10) \times U(1)]$. From the point of view of unification this is a very interesting coset space as both groups E_6 and $SO(10)$ are often used in grand unified theories. We follow the same steps as in the previous chapters where we discussed specific coset models. We first give the Kähler potential for the coset space in two very different forms. An alternative construction of the Kähler potential for this coset [129] uses the Kodaira's projective embedding theorem [38, 130]. We do not consider it here.

Next we discuss a few examples of matter coupling that we use in the construction of an anomaly-free model based on the fundamental representation $\underline{27}$ of E_6. We show that the charges of the section of the minimal line bundle and other matter representations are such that the charge assignment of anomaly cancellation can be achieved. We conclude with a short excursion to the phenomenology when we gauge all the isometries of the coset, that is the full E_6.

7.2 The $E_6/[SO(10) \times U(1)]$ Coset space

The algebra of E_6 can be written in terms of the following generators: M_{mn} are the generators of $SO(10)$, Y is the unbroken $U(1)$ generator commuting with $SO(10)$, and X_α and $\bar{X}^\alpha$ are the broken spinor generators, respectively. The indices $m, n = 1, \ldots, 10$ are the vector indices of $SO(10)$ and the indices $\alpha, \beta = 1, \ldots, 16$ are the chiral spinor indices. The coset is represented by the painted Dynkin diagram


. (7.1)

The generators satisfy the following commutation relations

$$\begin{aligned}
[Y, X_\alpha] &= \tfrac{1}{2}\sqrt{3}X_\alpha, & [Y, \bar{X}^\alpha] &= -\tfrac{1}{2}\sqrt{3}\bar{X}^\alpha,\\
[M_{mn}, X_\alpha] &= \tfrac{1}{2}i\Gamma^{+}_{mn\alpha}{}^{\beta}X_\beta, & [M_{mn}, \bar{X}^\alpha] &= -\tfrac{1}{2}i\bar{X}^\beta\Gamma^{+}_{mn\beta}{}^{\alpha},\\
[M_{mn}, M_{kl}] &= i\delta_{(mk}M_{nl)_c}, & [M_{mn}, Y] &= 0,\\
[X_\alpha, \bar{X}^\beta] &= \tfrac{1}{2}i\Gamma^{+}_{mn\alpha}{}^{\beta}M_{mn} + \sqrt{3}\delta_\alpha^\beta Y, & [X_\alpha, X_\beta] &= 0.
\end{aligned} \tag{7.2}$$

Here $(\dots)_c$ denotes cyclic permutation of the indices enclosed. Furthermore $\Gamma^+_{mn} = \Gamma_{mn}P^+$ is the positive chirality projection P^+ of the anti-symmetric product of the gamma matrices $\Gamma_{mn} = \frac{1}{2}[\Gamma_m, \Gamma_n]$. The matrices $\frac{1}{2}i\Gamma^+_{mn}$ form a spinor representation of the algebra satisfied by M_{mn}. The generators are taken Hermitean except for the spinor generators which obey $X^\dagger_\alpha = \bar{X}^\alpha$. On the fundamental $\underline{27}$ dimensional representation $\psi^T = (x, y_\alpha, x_m)$ of E_6, the generators are represented by the matrices [131]

$$\bar{\epsilon}\cdot X = \begin{pmatrix} 0 & \sqrt{2}\bar{\epsilon} & 0\\ 0 & 0 & \bar{\Sigma}\bar{\epsilon}^T\\ 0 & 0 & 0\end{pmatrix} \quad \bar{X}\cdot\epsilon = \begin{pmatrix} 0 & 0 & 0\\ \sqrt{2}\epsilon & 0 & 0\\ 0 & \epsilon^T\Sigma & 0\end{pmatrix} \tag{7.3}$$

$$\frac{1}{2}\omega\cdot M = -i\begin{pmatrix} 0 & 0 & 0\\ 0 & \frac{1}{4}\omega\cdot\Gamma^+ & 0\\ 0 & 0 & \omega\end{pmatrix} \quad \theta Y = \frac{1}{2\sqrt{3}}\begin{pmatrix} 4\theta & 0 & 0\\ 0 & \theta\mathbb{1}_{16} & 0\\ 0 & 0 & -2\theta\mathbb{1}_{10}\end{pmatrix}$$

Here we have used the sigma matrices $\Sigma_m = \Gamma_m P^+$ and $\bar{\Sigma}_m = P^+\Gamma_m$ in 10 dimensions. Here $\bar{\epsilon}^\alpha, \epsilon_\alpha$ are the infinitesimal parameters of the broken generators of E_6 and ω_{mn} and θ are the infinitesimal parameters of $SO(10)$ and $U(1)$, respectively. Using these matrices of the fundamental representation of E_6, the BKMU function $\xi(z)$ takes the form

$$\xi(z) = e^{f\,\bar{X}\cdot z} = \begin{pmatrix} 1 & 0 & 0\\ \sqrt{2}f\,z & \mathbb{1}_{16} & 0\\ \frac{1}{2}\sqrt{2}f^2\,z^T\Sigma z & f\,z^T\Sigma & \mathbb{1}_{10}\end{pmatrix} \tag{7.4}$$

where we have introduced a coupling constant f of mass dimension -1 and used that $P^+ = \mathbb{1}_{16}$ on the 16 component irreducible chiral spinor. The BKMU projection matrix η that projects on the one dimensional subspace of the $\underline{27}$ that has the highest Y weight is given by

$$\eta = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{7.5}$$

The Kähler potential for the coset space [131] can then be obtained using the methods of section 4.3

$$K_\eta = \frac{1}{f^2}\ln\left[1 + 2f^2\,\bar{z}z + \frac{1}{2}f^4(\bar{z}\bar{\Sigma}_m\bar{z}^T)(z^T\Sigma_m z)\right]. \tag{7.6}$$

By employing the "gauge fixing" method discussed in section 4.5 we find that another equivalent Kähler potential is given by

$$K_\sigma = \bar{z}Q^{-1}\ln(\mathbb{1}_{16} + Q)z = \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n} \bar{z}Q^{n-1}z, \tag{7.7}$$

using eqs. (4.87) and (4.104) with the redefinition of the parameter $f \to \sqrt{2}f$ and the algebra (7.2)

$$Q = \frac{1}{4} f^2 M^{\beta\delta}_{\alpha\gamma} z^\gamma \bar{z}_\delta \quad \text{and} \quad M^{\beta\delta}_{\alpha\gamma} = 3\delta^\beta_\alpha \delta^\delta_\gamma - \frac{1}{2}\Gamma_{mm\alpha}{}^{\beta}\Gamma_{mm\gamma}{}^{\delta}, \tag{7.8}$$

which was derived in ref. [32]. The relation between both seemingly different forms of the Kähler potential is clarified in ref. [117]. Using that the Kähler potential is uniquely defined up to a normalization factor when the coset contains just one $U(1)$ factor [104], we find that $K_\eta = 2K_\sigma$ by expanding to zeroth order in f^2. The tensor $M^{\beta\delta}_{\alpha\gamma}$ has the same symmetry properties as the tensor (4.87) defined in section 4.5. We find that $K_{\sigma,}{}^{\alpha} = [\bar{z}(\mathbb{1}_{16} + Q)^{-1}]^\alpha$ and $K_{\sigma,\alpha} = [(\mathbb{1}_{16} + Q)^{-1}z]_\alpha$ using those properties and hence the Killing potentials (2.18) are given by

$$\begin{aligned} M^\theta &= \frac{1}{f^2\sqrt{3}} - \frac{1}{2}\sqrt{3}\,\bar{z}^\alpha K_{\sigma,\alpha}, & M^{mn} &= -\tfrac{i}{2}\,\bar{z}^\alpha \Gamma^{+}_{mn\alpha}{}^{\beta} K_{\sigma,\beta}, \\ \bar{M}^\beta &= -\tfrac{1}{f}\,K_{\sigma,}{}^{\beta}, & M_\beta &= -\tfrac{1}{f}\,K_{\sigma,\beta}. \end{aligned} \tag{7.9}$$

The presence of the constant term in the $U(1)$ Killing potential M^θ is required by the closure of the Lie algebra on the Killing potentials. Having in hand the Kähler and Killing potentials, we can construct the holomorphic functions $F_i(z)$. A straightforward calculation yields

$$F^\theta = \frac{i}{f^2\sqrt{3}}, \quad F^{mn} = 0, \quad \bar{F}^\beta = 0, \quad F_\beta = -\frac{i}{f}\,z_\beta. \tag{7.10}$$

7.3 Matter Coupling to $E_6/[SO(10) \times U(1)]$

We now describe how matter representations can be obtained from the tangent bundle and line bundle. The infinitesimal transformation rules for a tangent vector T_α of the coset, a spinor of $SO(10)$, is given by

$$\delta T_\alpha = \left(\frac{i}{2}\sqrt{3}\,\theta\delta^\beta_\alpha - \frac{1}{4}\omega_{mn}\Gamma^{+}_{mn\alpha}{}^{\beta} - \frac{i}{4} f\bar{\epsilon}^\gamma z_\delta M^{\delta\beta}_{\gamma\alpha} \right) T_\beta. \tag{7.11}$$

The vector V^m of $SO(10)$, a specific section of the tangent bundle squared, transforms as [69]

$$\delta V^m = \left(i\sqrt{3}\,\theta\delta_{mn} - \omega_{mn} - if\,\bar{\epsilon}\big(\Gamma^{+}_{mn} + 3\delta_{mn}P^{+}\big)z \right) V^n. \tag{7.12}$$

To obtain a section of the minimal line bundle, we employ the methods developed in subsection 4.3.2. The infinitesimal transformation of this singlet L takes the form

$$\delta L = 2\left(\frac{i}{\sqrt{3}}\,\theta - i\,f\bar{\epsilon}\cdot z\right)L, \tag{7.13}$$

using the transformation rule (3.14) and the holomorphic functions (7.10). The factor 2 ensures that L is a section of the minimal line bundle, as can be seen in two ways. Both ways rely on the fact that the $\underline{27}$ of E_6 is the representation with all its Dynkin labels zero except for the first one and it therefore identifies the minimal line bundle. Using the matrix expression for Y in the fundamental representation of E_6, we see that the charge of a section L of the minimal line bundle is equal to 4/3 of the charge of the coordinates of the coset space $E_6/[SO(10)\times U(1)]$. With the factor of 2 in (7.13) the appropriate relative charge is obtained. Equivalently the BKMU Kähler potential K_η transforms into the holomorphic functions of the minimal line bundle. Since $K_\eta = 2K_\sigma$ we obtain the same result for the relative charges. The metrics for the matter representations T, V, L are given by

$$g_T = (1\!\!1_{16} + Q)^{-2}\,, \qquad g_L = e^{-f^2K_\eta} = e^{-2f^2K_\sigma}, \tag{7.14}$$

$$g_{V\,mn} = -\frac{1}{16}g_{T}{}_\alpha^{\ \beta} g_{T}{}_\gamma^{\ \delta}\,\left(C\bar{\Sigma}_m\right)^{\alpha\gamma}(\Sigma_n C)_{\beta\delta} = \frac{1}{16}\mathrm{tr}\left(g_T\big(\Sigma_m C\big)^\dagger g_T\big(\Sigma_n C\big)\right).$$

The factor $\frac{1}{16}$ is introduced to normalize the metric $g_{V\,mn}$ to δ_{mn} in the limit $z = 0$. The charge conjugation matrix of $SO(10)$ is denoted by C.

We now discuss the extension of the $E_6/[SO(10)\times U(1)]$ coset model to a supersymmetric model with chiral superfields in the $\underline{27}$ of E_6. The $\underline{27}$ branches to $\underline{1}(4) + \underline{16}(1) + \underline{10}(-2)$ representation of $SO(10)\times U(1)$. From the fourth column in table 7.1 we can read off in which tensor-products of bundles the complex matter scalars N_m and h reside. The tangent bundle T is used when the matter representation has a $SO(10)$ spinor index and when the matter representation the bundle of the $SO(10)$ vector V is used; the number of spinor and vector indices determines the powers of T and V. The rescaling charge, the power of the line bundle L, is determined as follows. Consider a matter representation that sits in T^rV^s and in the qth power of the line bundle , its charge relative to the charge of the coordinates is $Y = r + 2\,s + \frac{4}{3}\,q$. We see that N^m has a rescaling charge $q = -3$ while the physical singlet representation h has a rescaling charge $q = 3$. The scalar components we discussed thus far can be embedded in the chiral multiplets $\Phi_\alpha = (z_\alpha, \psi_{L\,\alpha})$, $\Psi_m = (N_m, \chi_{L\,m})$ and $\Omega = (h, \chi_L)$.

An invariant kinetic action for these matter superfields is obtained from the Kähler potential

$$K_m \;=\; e^{-6f^2K_\sigma(\bar{\Phi},\Phi)}\,\bar{\Omega}\Omega \;+\; e^{6f^2K_\sigma(\bar{\Phi},\Phi)}\,g_{mn}(\bar{\Phi},\Phi)\,\bar{\Psi}_m\Psi_n, \tag{7.15}$$

Dimension repr.	$U(1)$ charge	Notation	Tensor bundle	Description of the type of fields
16	1	z_α		$E_6/[SO(10)\times U(1)]$ coset coordinates
10	-2	N^m	VL^{-3}	Matter additions to <u>16</u>
1	4	h	L^3	to complete the <u>27</u>

Table 7.1: The $SO(10)$ representations used for our construction of an anomaly-free model based on $E_6/[SO(10)\times U(1)]$ with the field content of the <u>27</u> of E_6. The first column gives the dimension of the representations, the second column their charges, the fourth column the notation we use for the scalar components of chiral multiplets. In the third column we have indicated how the various matter representations form tensor products of the tangent bundle (T), the bundle (V) of the vector of $SO(10)$ and the minimal line bundle (L). A brief description of what these fields are is given in the last column.

where we have chosen to work with K_σ in stead of K_η. It remains to construct the extended Killing potentials for the coset space plus the matter coupling

$$\begin{aligned}\mathcal{M}_i =& M_i \left(1-6f^2 e^{-6f^2K_\sigma}|h|^2+6f^2e^{6f^2K_\sigma}g_{V\,mn}\bar{N}_m N_n\right)\\ &-\frac{1}{8}e^{6f^2K_\sigma}M_{i,\alpha}{}^{\beta}\,g_{T}{}_{\gamma}^{\delta}\left(C\bar{\Sigma}_m\right)^{\alpha\gamma}\left(\Sigma_n C\right)_{\beta\delta}\bar{N}_m N_n,\end{aligned} \tag{7.16}$$

with the Killing potentials M_i given by the expressions (7.9). In the case the full E_6 symmetry is gauged with gauge coupling g, the scalar potential is given by

$$V_{unitary} = \frac{g^2}{2}\sum_i\left[\mathcal{M}_i(z=\bar{z}=0;\bar{h},h;\bar{N}_m,N_m)\right]^2 \tag{7.17}$$

$$= \frac{g^2}{2}\left(\frac{1}{f^2\sqrt{3}}-2\sqrt{3}|h|^2+\sqrt{3}\sum_m|N_m|^2\right)^2+\frac{g^2}{2}\sum_{m,n}|\bar{N}_m N_n-\bar{N}_n N_m|^2.$$

Here we have used that all Goldstone bosons disappear from the spectrum as a result of the Brout-Englert-Higgs effect in the unitary gauge $z_\alpha=0$. The set of supersymmetric minima is characterized by the equations

$$|h|^2=\frac{1}{6f^2}+\frac{1}{2}\sum_k|N_k|^2,\quad |\bar{N}_m N_n-\bar{N}_n N_m|^2=0 \tag{7.18}$$

for all m,n. It follows that $|h|\neq 0$ and the $U(1)$ gauge symmetry is always broken; a solution with $N_m=0$ is possible, preserving $SO(10)$. However, solutions with

$N_m \neq 0$ breaking $SO(10)$ are allowed and expected in a next stage of symmetry breaking. If we write $N_m = |N_m|e^{i\phi_m}$, we can get access to the structure of the $SO(10)$ breaking vacua because the second set of equations in (7.18) becomes

$$|N_m|^2|N_n|^2 \sin^2(\phi_n - \phi_m) = 0. \tag{7.19}$$

Therefore a component N_m of vector N is either zero or its phase ϕ_m is equal modulo π to a common phase ϕ of all non-vanishing components of the vector N:

$$N_m = 0 \quad \text{or} \quad \phi_m = \phi \mod \pi. \tag{7.20}$$

The absolute values $|N_m|$ of the non-vanishing components of the vector N and the common phase ϕ can be interpreted as moduli fields, since they do not have to be constants and they characterize different supersymmetric vacua.

Chapter 8

Conclusions

Supersymmetric models based on Kähler manifolds often have a fermionic field content which makes the isometries of these manifolds anomalous. There are several methods available to remove these anomalies. One of them is to add additional fermions that reside in chiral multiplets such that all anomalies cancel. These matter multiplets have to be sections of bundles over the original manifold such that the full model is again based on a Kähler manifold. One needs a Kähler manifold to ensure that one can use it for supersymmetric model building. For any given Kähler manifold the following program has been put forward to turn it into an anomaly-free supersymmetric model: first, investigate what kind of matter couplings can be constructed and are well-defined globally. Next, choose an anomaly-free representation of the isometry group and check whether it is possible to obtain the right matter multiplets to construct this representation. In this context, the non-trivial singlet plays a very important role as it makes flexible choices for the $U(1)$ charges of the matter multiplets possible. If this singlet is to exist globally, it has to be a section of a line bundle. Its charge is quantized if the target space has a non-trivial topology because the consistency conditions of the line bundle are not automatically satisfied. We discussed the coupling of a supersymmetric σ-model with additional matter multiplets to supergravity. The Weyl weights of these matter multiplets can be interpreted as rescaling charges.

A large part of this thesis has been devoted to the study of supersymmetric models based on Kählerian coset spaces G/H. We have discussed various methods to obtain the Kähler potentials for those spaces. For a large class of coset spaces we studied their isometry properties in both infinitesimal and finite form. To investigate the local geometrical structure of cosets, the Killing vectors provide the necessary information: the metric and other geometrical quantities can be derived from them. In particular, there are various methods to obtain the Kähler potential from the Killing vectors. The Killing vectors are also essential to construct the covariant derivatives for the gauged isometries. One can check global properties using the finite form of the isometries; because a Kählerian coset is homogeneous, each point on it can be reached by a group transformation. Hence one can use group properties to check whether a matter representation has the interpretation of a section of a bundle.

Even though the restriction posed by the consistency of the line bundle for a non-trivial singlet is quite severe, we have constructed several anomaly-free models. Let us list these models. Based on the Grassmannian coset space $SU(5)/[SU(2) \times U(1) \times SU(3)]$ we constructed a version of the standard model which contains a non-linear $SU(5)$ symmetry. This model has the field content of one generation of quarks and leptons (the inclusion of more generations is not difficult) and the two Higgs-doublets required by supersymmetry. We have also constructed a similar model based on the non-compact version $SU(2,3)$ of $SU(5)$; we needed a different covariant superpotential in that case. The anomaly-free model based on the coset $E_6/SO(10) \times U(1)$ has the representation structure determined by the $SO(10) \times U(1)$ branching of the $\underline{27}$ of E_6. The coordinates of the coset are identified with the $\underline{16}$ of $SO(10)$. The $\underline{1}$ and the $\underline{10}$ are obtained as sections of tensor products of the line and tangent bundles, respectively. Judging from these two examples one might expect that a consistent model based on a Kählerian coset can always be obtained. However, we have shown that only a finite number of models can be constructed based on the coset spaces $SO(2N)/U(N)$ with $U(N)$ matter representations which are determined by the spinor representation of $SO(2N)$. The line bundle consistency constraint can only be satisfied for the cases $N = 2, 5, 6, 8$. Isometry anomalies are absent because for $N \neq 1, 3$ all representations of $SO(2N)$ are anomaly-free.

The phenomenology has been studied for three cases: the Grassmannian coset $SU(5)/[SU(2) \times U(1) \times SU(3)]$, $E_6/SO(10) \times U(1)$ and $SO(10)/U(5)$. In each of these models we tried to interpret the coordinates as part of super multiplets describing one generation of quarks and leptons. Because the coordinate transformation is non-linear under the broken isometries, they cannot be included in the superpotential. We have studied the coset $E_6/SO(10) \times U(1)$ and $SO(10)/U(5)$ in the context of global supersymmetry. It turned out that there exist supersymmetric minima for both these models when the full isometry groups E_6 and $SO(10)$ are gauged. In both cases we found that the factor $U(1)$ of the linear subgroup is spontaneously broken. The Grassmannian coset $SU(5)/[SU(2) \times U(1) \times SU(3)]$ with the supersymmetric standard model field content was studied in the context of local supersymmetry. The superpotential transforms covariantly under the $SU(5)$ isometries because the Kähler potential is also covariant. In particular, it transforms under the $U(1)$ that is interpreted as the hyper-charge $U_Y(1)$ of the standard model. Therefore the superpotential is not allowed to acquire a vacuum expectation value, for else the hyper-charge $U_Y(1)$ and hence the electromagnetic $U_{em}(1)$ are broken. As a consequence we found that the scalar potential arises from D-terms only. Its minimum breaks supersymmetry and we obtain a prediction for $\tan\beta$, the ratio of the vacuum expectation values of the Higgses.

We conclude with an outlook of possible further developments in this field. A complete classification of supersymmetric models based on (Kählerian) cosets should be obtained. In this thesis we focused on the possibility of the cancellation of all isometry anomalies of homogeneous cosets only. For non-homogeneous cosets anomalies can also be removed by a Wess-Zumino counter term. The phenomenology of various models could be worked out in more detail: the inclusion of Higgs-sectors and three generations of quarks and leptons could be analyzed. The mass spectrum and other possible physical observables have to be investigated. Furthermore, these models could be analyzed in the context of both global and local supersymmetry. We have restricted the applications of models based on cosets to standard model unification only. There is no need to do this; these models could also provide us with an effective description of a hidden sector. Furthermore, the moduli spaces in string theory show many similarities with the models under investigation in this thesis. There is one big difference though: we have studied continuous (often infinitesimal) symmetries, while the global symmetries acting on moduli spaces are discrete. As the models we discussed could be valid up to the Planck scale when studied in the context of supergravity, they may also be relevant for astro-physical processes like inflation [132].

Samenvatting

Alle materie die we om ons heen zien, is opgebouwd uit elektronen, protonen en neutronen. Voor zover we nu weten, zijn elektronen elementaire deeltjes. Dat betekent dat we nog geen substructuur hebben kunnen ontdekken in deze deeltjes. Aan de andere kant zijn protonen en neutronen opgebouwd uit quarks. De quarks en elektronen kunnen met elkaar wisselwerken via verschillende krachten, te weten de elektromagnetische, de sterke en de zwakke interacties. De namen de “sterke” en de “zwakke” kracht slaan in eerste instantie op hun relatieve koppelingsterkte (kracht die de deeltjes op elkaar uitoefenen), maar er worden weldegelijk twee verschillende krachten mee bedoelt. De elektromagnetische kracht bindt elektronen en atoomkernen aan elkaar zodat ze atomen vormen. De sterke kracht sluit de quarks binnen protonen en neutronen op en zorgt er op een indirecte wijze ook voor dat atoomkernen niet uit elkaar vallen. De zwakke kracht is verantwoordelijk voor verval van bepaalde atoomkernen. Naast deze drie krachten is er natuurlijk nog de zwaartekracht. Maar de zwaartekracht is zo zwak, dat die voor subatomaire processen te verwaarlozen is. Het huidige model, dat een beschrijving van de natuur in term van deze drie krachten levert, wordt het “standaardmodel” genoemd.

De krachten in het standaardmodel worden overgebracht door deeltjes die uitgewisseld worden. Fotonen waar licht uit bestaat zijn de bekendste wisselwerkingsdeeltjes. De sterke kracht heeft acht wisselwerkingsdeeltjes die gluonen worden genoemd. Alleen de drie krachtoverbrengers van de zwakke interactie, de W^+, W^- en Z bosonen, blijken een massa te hebben. Naast deze geobserveerde deeltjes wordt het bestaan van het Higgs-boson verondersteld. Dit deeltje is noodzakelijk om de andere deeltjes hun massa’s te geven. (Hoewel ook het Higgs-boson een interactie overdraagt, wordt hij op een totaal andere manier beschreven in de theorie. Daarom wordt de Higgs-interactie niet in het rijtje van krachten geplaatst.) Ook het Higgs-boson maakt deel uit van het standaardmodel.

Het is niet zo makkelijk om grip op de beschrijving van deze wisselwerkingen te krijgen. Met symmetrieën kunnen de eigenschappen van de theorieën, die de krachten beschrijven, gekarakteriseerd worden. Dit is erg efficiënt: in de plaats van alle eigenschappen op te geven, is het voldoende om de symmetrie aan te geven waaruit dat hele lijstje eigenschappen volgt. Een symmetrie is een verandering in de beschrijving die tot dezelfde fysische conclusies leidt. Om dit wat concreter te maken denk bijvoorbeeld aan een ronde bal. Als die geroteerd wordt, verandert er niets aan de eigenschappen van de bal. Merk op dat het niet uit maakt om welke as door het midden van de bal geroteerd wordt. De bal voor en na de rotatie zijn niet onderscheidbaar, dus de rotatie is een symmetrie van de bal. De rotatie-invariantie om een willekeurige as is één van de karakteristieke kernmerken van de bal, immers hiermee is de ronde bal te onderscheiden van een rugbybal; deze is alleen maar invariant onder een willekeurige rotatie om zijn lengteas.

Op een vergelijkbare wijze worden symmetrieën in de elementaire-deeltjesfysica gebruikt om de eigenschappen van theorieën vast te leggen. Door naar de bal en de rugbybal te kijken, is het duidelijk dat het twee verschillende objecten zijn, maar om het verschil tussen beide objecten te karakteriseren is een symmetrybeschouwing eigenlijk de enige mogelijkheid. We kunnen niet direct naar een theorie kijken, maar alleen haar gevolgen onderzoeken. Daarom is een symmetriebeschouwing een redmiddel om de eigenschappen van een theorie van elementaire deeltjes vast te leggen.

De ultieme test van een theoretische beschrijving van de materie is natuurlijk of de voorspellingen van de theorie door experimenten te verifiëren zijn. Het standaardmodel is in overeenstemming gebleken met alle metingen die hebben plaatsgevonden bij hoge-energieëxperimenten. Ondanks dit enorme succes, is men niet echt tevreden met dit model. Het lijkt aan elkaar te hangen van allemaal tamelijk willekeurige keuzen, die te toevallig lijken te zijn om echt het laatste woord te zijn. Anders gezegd, we vinden de uitgangspunten van het standaardmodel veel te ad hoc. Twee belangrijke en veelvuldig bestudeerde ideeën om meer systematiek in de fundamenten van het standaardmodel aan te brengen zijn unificatie en supersymmetrie. Aangezien deze concepten ook ten grondslag liggen aan dit proefschrift, zullen we ze even kort toegelichten.

In een unificatietheorie gaat men ervan uit dat de sterke, zwakke en elektromagnetische krachten in het standaardmodel eigenlijk verschillende manifestaties zijn van één overkoepelende interactie. Dit betekent bijvoorbeeld dat de koppelingssterkten van de krachten aan elkaar gerelateerd zijn. Door de gemeten energieafhankelijkheid van deze koppelingsconstanten te extrapoleren, blijkt dat er een energiegebied is waar de koppelingen ongeveer gelijk zijn. Een ander gevolg blijk te zijn dat deeltjes, die in eerste instantie weinig overeenkomsten lijken te hebben zoals elektronen en quarks, als twee kanten van dezelfde munt zijn.

Om het begrip supersymmetrie toe te lichten, is het handig om de standaardmodel deeltjes in te delen in "sociale" en "asociale" deeltjes. Sociale deeltjes zitten graag allemaal in exact dezelfde toestand, terwijl de asociale deeltjes ieder een toestand voor zichzelf alleen opeisen. De sociale deeltjes heten bosonen en die blijken in het standaardmodel de wisselwerkingsdeeltjes te zijn. De asociale deeltjes worden fermionen genoemd en zijn de bouwstenen van de materie om ons heen, namelijk de elektronen en quarks. Het blijkt mogelijk te zijn om aan het standaardmodel een aantal deeltjes toe te voegen zodat ieder fermion een bosonische partner heeft en andersom. Dit ziet er niet alleen esthetischer uit, maar het heeft ook een andere belangrijke eigenschap: er treden geen kwadratische oneindigheden op. Deze oneindigheden, ontploffingen in berekeningen, kunnen verschillende sterkten hebben. Een "logaritmische" divergentie gedraagd zich als een rotje, terwijl een "kwadratische" divergentie veel weg heeft van een exploderend stuk dynamiet. Dankzij het werk van onze Nederlandse Nobelprijswinnaars G. 't Hooft en M. Veltman weten we hoe we door dit mijnenveld heen kunnen wandelen. In het gewone standaardmodel treden er wel kwadratische divergenties op en die leiden tot het vervelende gevolg dat de Higgs-massa kwadratische afhankelijk is van massa's van deeltjes die veel zwaarder zijn dan de Higgs zelf. In het supersymmetrische standaardmodel is de afhankelijkheid van de massa's van zware en tot nu toe onbekende deeltjes veel minder sterk. De reden is dat de supersymmetrische variant van het standaardmodel alleen logaritmische divergenties bevat, het geen aanleiding geeft tot slechts logaritmische gevoeligheid op de massa's van deze niet waargenomen deeltjes.

Het raamwerk waarbinnen de deeltjes van het standaardmodel of een (supersymmetrische) unificatietheorie beschreven worden, is de zogenaamde veldentheorie. In een veldentheorie worden de eigenschappen beschreven in termen van velden, functies van tijd en ruimte. Kleine trillingen van velden, zogenaamde kwantumexcitaties, bewegen zich als deeltjes voort. Het is echter niet uitgesloten dat die velden niet alle reële waarden kunnen aannemen, maar beperkt worden tot bijvoorbeeld een interval. Een supersymmetrische theorie kan bijvoorbeeld op een bol leven in de plaats van de volledige tweedimensionale ruimte. Het is niet direct duidelijk of alle eigenschappen van een gewone veldentheorie, zoals het standaardmodel, over te dragen zijn. In dit proefschrift is onderzocht in hoeverre het mogelijk is bij een willekeurig hoger dimensionaal gekromd oppervlak een supersymmetrische veldentheorie te associëren en welke eigenschappen die heeft.

Anomalieën kunnen een problematische hindernis zijn om op een gegeven gekromd oppervlak een supersymmetrische veldentheorie te construeren. Anomalieën zijn eigenschappen of symmetrieën die door kwantumeffecten vernietigd worden. Sommige anomale symmetrieën hebben catastrofale gevolgen: de theorie is inconsistent en stort als een plumpudding in elkaar.

In dit proefschrift hebben we in deze zin gevaarlijke supersymmetrische theorieen opgezocht en laten zien dat we om deze diepe afgronden heen kunnen laveren door ze uit te breiden met extra supersymmetrische velden. Deze uitbreidingen kunnen soms zelf ook weer problemen opwerpen, vooral als de supersymmetrische theorie gebaseerd is op een topologisch niet-triviaal oppervlak, zoals een tweedimensionale bol. Met een topologisch niet-triviaal oppervlak wordt bedoeld dat het niet mogelijk is het oppervlak tot een oneindig uitgestrekt plat vlak te vervormen zonder het oppervlak te scheuren. Hieraan voldoet de tweedimensionale bol, omdat alleen door er een gat in te knippen en dan uit te rekken het op een plat vlak gaat lijken. Meer wiskundig gezegd is het probleem bij dit soort oppervlakken dat niet mogelijk is, ze met één set coördinaten te beschrijven. Een bol kan bijvoorbeeld met twee sets van coördinaten overdekt worden: een van de noordpool tot de evenaar en een van de zuidpool tot de evenaar. Deze twee halfbollen kunnen natuurlijk wel door wat uitrekken tot platte vlakken gemaakt worden.

Terug naar het voorbeeld van de supersymmetrische theorie op een tweedimensionale bol. Zoals eerder opgemerkt is de bol rotatie-symmetrisch. Bij de supersymmetrische kwantumveldentheorie gebaseerd op de bol is deze rotatiesymmetrie geschonden, we hebben te maken met een anomalie. We willen het model op een supersymmetrische manier uitbreiden zodanig dat de anomalie verdwijnt. Maar omdat de bol topologisch niet-triviaal is, moeten we wel oppassen dat deze uitbreiding zich verdraagt met overgang van de beschrijvingen op het noordelijk en zuidelijk halfrond.

Naast de constructie van uitbreidingen van supersymmetrische modellen gebaseerd op gekromde oppervlakken, zijn ook wat fenomenologische (mogelijk observeerbare) gevolgen van dit soort theorieën bestudeerd. Een veldentheorie beschrijft deeltjes, dus kunnen eigenschappen als lading en massa van die deeltjes bestudeerd worden. Een andere interessante fenomenologische vraag is of de rusttoestand van de velden, hun zogenaamde vacuüm verwachtingswaarden wel alle symmetrieën respecteren. (Dit is te zien door op zo'n toestand een symmetrie transformatie los te laten en kijken of deze toestand invariant blijft.) Als dat niet het geval is, dan is de natuurlijke vraag welke gevolgen dat heeft. Dit kan bijvoorbeeld zijn het ontstaan van massa voor sommige deeltjes of schendig van bepaalde behoudswetten.

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Appendix A

Complex Differential Geometry

This appendix provides the mathematical background for the geometrical discussion of supersymmetric σ-models as given in chapter 2. For a more detailed and complete discussion on complex differential geometry the reader may consult [37, 38] or for more physical intuition [18, 39].

We start by defining complex manifolds and complex functions on them. A complex manifold $\mathfrak{M}$ is a topological space with an open covering $\{U_{(a)}\}$, with the following properties. On each open patch $U_{(a)}$ an invertible coordinate function $\phi^{(a)}$ is defined which maps $U_{(a)}$ into $\phi_{(a)}(U_{(a)}) \subset \mathbb{C}^n$. The local coordinates of the image $\phi_{(a)}(U_{(a)}) \subset \mathbb{C}^n$ we denote by $Z^{\mathcal{A}}_{(a)}$, although for readability we often write $Z^{\mathcal{A}}$ when no confusion is possible. For a non-empty intersection $U_{(a)} \cap U_{(b)}$ the transition function

$$\phi_{(ba)} = \phi_{(b)} \circ \phi^{-1}_{(a)} : \phi_{(a)}(U_{(a)} \cap U_{(b)}) \subset \mathbb{C}^n \longrightarrow \mathbb{C}^n \tag{A.1}$$

is analytic. In other words the coordinate transformation $Z_{(a)} \longrightarrow Z_{(b)}(Z_{(a)})$ is holomorphic. Around any point $p \in \mathfrak{M}$ the manifold looks like a subset of $\mathbb{C}^n$.

To define a complex function $F : \mathfrak{M} \longrightarrow \mathbb{C}$ on the manifold $\mathfrak{M}$, we first take complex functions $F_{(a)} : \phi_{(a)}(U_{(a)}) \longrightarrow \mathbb{C}$ locally on the different patches $U_{(a)}$. When these functions are compatible in the sense that, on a non-void intersection of two coordinate patches the corresponding two functions $F_{(a)}$ and $F_{(b)}$ are related to each other as

$$F_{(a)} = F_{(b)} \circ \phi_{(ba)}, \tag{A.2}$$

the function F can be defined unambiguously by $F \equiv F_{(a)} \circ \phi_{(a)}$.

Next we discuss how a complex structure is introduced and how a Hermitean metric can be defined on a complex manifold. At each point $p \in \mathfrak{M}$ a tangent space $T_p\mathfrak{M}$ is defined as the vector space generated by the basis $(\partial_{\mathcal{A}})_p = (\frac{\partial}{\partial Z^{\mathcal{A}}})_p$ and $(\bar{\partial}_{\underline{\mathcal{A}}})_p = (\frac{\partial}{\partial \bar{Z}^{\underline{\mathcal{A}}}})_p$: $T_p\mathfrak{M} = \left\{ V^{\mathcal{A}}(\partial_{\mathcal{A}})_p + \bar{V}^{\underline{\mathcal{A}}}(\bar{\partial}_{\underline{\mathcal{A}}})_p | V^{\mathcal{A}}, \bar{V}^{\underline{\mathcal{A}}} \in \mathbb{C} \right\}$. The cotangent space $T^*_p\mathfrak{M} = \left\{ V_{\mathcal{A}}(dZ^{\mathcal{A}})_p + \bar{V}_{\underline{\mathcal{A}}}(d\bar{Z}^{\underline{\mathcal{A}}})_p | V_{\mathcal{A}}, \bar{V}_{\underline{\mathcal{A}}} \in \mathbb{C} \right\}$ is the space of linear operators on the tangent space $T_p\mathfrak{M}$ and is therefore the dual space of $T_p\mathfrak{M}$. The basis of the cotangent space $T_p^*\mathfrak{M}$ are the differentials $dZ^{\mathcal{A}}, d\bar{Z}^{\underline{\mathcal{A}}}$, that satisfy

$$dZ^{\mathcal{B}}(\partial_{\mathcal{A}}) = \delta^{\mathcal{B}}{}_{\mathcal{A}}, \quad d\bar{Z}^{\underline{\mathcal{B}}}(\bar{\partial}_{\underline{\mathcal{A}}}) = \delta_{\underline{\mathcal{A}}}{}^{\underline{\mathcal{B}}} \quad \text{and} \quad d\bar{Z}^{\underline{\mathcal{B}}}(\partial_{\mathcal{A}}) = dZ^{\mathcal{B}}(\bar{\partial}_{\underline{\mathcal{A}}}) = 0. \tag{A.3}$$

Under a change of coordinates (A.1) the differentials and derivatives transform as

$$\begin{aligned} dZ_{(b)} &= X_{(ba)} dZ_{(a)}, & d\bar{Z}_{(b)} &= d\bar{Z}_{(a)} \bar{X}_{(ba)}, \\ \partial_{(b)} &= \partial_{(a)} X^{-1}_{(ba)}, & \bar{\partial}_{(b)} &= \bar{X}^{-1}_{(ba)} \bar{\partial}_{(a)}, \end{aligned} \tag{A.4}$$

with the transformation matrices

$$\left(X_{(ba)}\right)^{\mathcal{B}}{}_{\mathcal{A}} = \frac{\partial Z^{\mathcal{B}}_{(b)}}{\partial Z^{\mathcal{A}}_{(a)}}, \quad \text{and} \quad \left(\bar{X}_{(ba)}\right)_{\underline{\mathcal{A}}}{}^{\underline{\mathcal{B}}} = \frac{\partial \bar{Z}^{\underline{\mathcal{B}}}_{(b)}}{\partial \bar{Z}^{\underline{\mathcal{A}}}_{(a)}}. \tag{A.5}$$

The tangent and cotangent bundles $T\mathfrak{M}$ and $T^*\mathfrak{M}$ are defined as the disjunct union at each point $p \in \mathfrak{M}$ of the tangent $T_p\mathfrak{M}$ and cotangent $T_p^*\mathfrak{M}$ spaces, respectively. Bundles are discussed in a bit more detail in appendix B, although there the focus is on bundles over Kähler manifolds.

On a complex manifold a complex structure J can be defined as a constant section of the tensor bundle $T^*\mathfrak{M} \otimes T\mathfrak{M}$, using local coordinates

$$J = i\left(dZ^{\mathcal{A}} \otimes \partial_{\mathcal{A}} - d\bar{Z}^{\underline{\mathcal{A}}} \otimes \partial_{\underline{\mathcal{A}}}\right). \tag{A.6}$$

It follows directly from the transformation properties of differentials and derivatives (A.4) that J is indeed a constant tensor field on the whole manifold $\mathfrak{M}$.

A symmetric tensor field G of $T^*\mathfrak{M} \otimes T^*\mathfrak{M}$, which acts as a bilinear operator from $T\mathfrak{M} \otimes T\mathfrak{M} \longrightarrow \mathbb{C}$, is called a metric. In local coordinates the metric acting on the vectors V and W takes the form

$$G(V,W) = \begin{pmatrix} V^{\mathcal{A}} & \bar{V}^{\underline{\mathcal{A}}} \end{pmatrix} \begin{pmatrix} G_{\mathcal{A}\mathcal{B}} & G_{\mathcal{A}\underline{\mathcal{B}}} \\ G_{\underline{\mathcal{A}}\mathcal{B}} & G_{\underline{\mathcal{A}}\underline{\mathcal{B}}} \end{pmatrix} \begin{pmatrix} W^{\mathcal{B}} \\ \bar{W}^{\underline{\mathcal{B}}} \end{pmatrix}. \tag{A.7}$$

From the definitions it is clear that $G(V,W)$ is a coordinate independent quantity. We take this metric to be Hermitean, that is for all $V, W \in T_p\mathfrak{M}$ we have that

$$G(JV, JW) = G(V,W) \quad \text{or} \quad G_{\underline{\mathcal{A}}\,\underline{\mathcal{B}}} = G_{\mathcal{A}\mathcal{B}} = 0, \quad G_{\underline{\mathcal{A}}\mathcal{B}} = G_{\mathcal{A}\underline{\mathcal{B}}}. \tag{A.8}$$

Under a change of coordinates (A.1) the components of a Hermitean metric transform as

$$G_{(b)} = \bar{X}^{-1}_{(ba)} G_{(a)} X^{-1}_{(ba)} \quad \text{or} \quad G_{\underline{\mathcal{A}}\mathcal{A}} = G_{\underline{\mathcal{B}}\mathcal{B}} \frac{\partial \bar{Z}^{\underline{\mathcal{B}}}}{\partial \bar{Z}^{\underline{\mathcal{A}}}} \frac{\partial Z^{\mathcal{B}}}{\partial Z^{\mathcal{A}}}. \tag{A.9}$$

The Kähler $(1,1)$-form ω is defined as

$$\omega(V,W) = G(JV,W), \qquad \omega = -iG_{\underline{\mathcal{A}}\mathcal{A}} d\bar{Z}^{\underline{\mathcal{A}}} \wedge dZ^{\mathcal{A}}, \tag{A.10}$$

where the wedge product $d\bar{Z}^{\underline{\mathcal{A}}} \wedge dZ^{\mathcal{A}}$ is defined by

$$d\bar{Z}^{\underline{\mathcal{A}}} \wedge dZ^{\mathcal{A}} = d\bar{Z}^{\underline{\mathcal{A}}} \otimes dZ^{\mathcal{A}} - dZ^{\mathcal{A}} \otimes \bar{Z}^{\underline{\mathcal{A}}}. \tag{A.11}$$

On a complex manifold three types of exterior derivatives are defined: two complex ones $\boldsymbol{\partial}$ and $\bar{\boldsymbol{\partial}}$ and a real one $\mathbf{d}$, which are given by

$$\boldsymbol{\partial} = \partial_{\mathcal{A}} dZ^{\mathcal{A}} \wedge, \quad \bar{\boldsymbol{\partial}} = \bar{\partial}_{\underline{\mathcal{A}}} d\bar{Z}^{\underline{\mathcal{A}}} \wedge \quad \text{and} \quad \mathbf{d} = \boldsymbol{\partial} + \bar{\boldsymbol{\partial}}. \tag{A.12}$$

If the Kähler form ω is closed

$$\mathbf{d}\omega = 0 \quad \text{or} \quad G_{\underline{\mathcal{A}}\mathcal{A},\mathcal{B}} = G_{\underline{\mathcal{A}}\mathcal{B},\mathcal{A}}, \quad G_{\underline{\mathcal{A}}\mathcal{B},\underline{\mathcal{B}}} = G_{\underline{\mathcal{B}}\mathcal{A},\underline{\mathcal{A}}}, \tag{A.13}$$

the manifold is called a Kähler manifold. This definition is equivalent to the statement that in local coordinates the Kähler metric $G_{\underline{\mathcal{A}}\mathcal{A}}$ can be obtained as the second mixed derivative

$$G_{\underline{\mathcal{A}}\mathcal{A}} = \mathcal{K}_{,\underline{\mathcal{A}}\mathcal{A}} \tag{A.14}$$

of a real symmetric function $\mathcal{K}(\bar{Z}, Z)$ called the Kähler potential. The Kähler potential is not a function on the manifold. Indeed, only the metric obtained from the Kähler potential by taking the second mixed derivative (A.14) is form invariant (A.9). On the overlap of two coordinate patches $U_{(a)}$ and $U_{(b)}$ the Kähler potentials defined on both patches are related to each other via

$$\mathcal{K}_{(a)} = \mathcal{K}_{(b)} + \mathcal{F}_{(ab)} + \bar{\mathcal{F}}_{(ab)}, \tag{A.15}$$

where $(\bar{\mathcal{F}}_{(ab)})\mathcal{F}_{(ab)}$ is an (anti)-holomorphic function. The Kähler form can therefore be written as

$$\omega(\mathcal{K}) = -i\bar{\boldsymbol{\partial}}\boldsymbol{\partial}\mathcal{K} \tag{A.16}$$

globally.

Appendix B

Bundles over Kähler Manifolds

The formalism of bundles over Kähler manifolds provides the mathematically correct language to discuss matter coupling to non-linear supersymmetric σ-models. We first give a general review of fibre bundles and after that we discuss several examples of bundles which we have used to obtain matter representations for supersymmetric model building in section 3.3, starting with the tangent and cotangent bundles. The line bundle is discussed in more detail as its consistency has serious consequences for matter coupling when the manifold $\mathfrak{M}_\sigma$ is compact. This appendix is largely based on the mathematical literature [37, 38, 39, 133].

Consider a fibre bundle $\pi : E \longrightarrow \mathfrak{M}_\sigma$ with π a continuous projection on the base space $\mathfrak{M}_\sigma$. The local triviality of the fibre bundle tells us that locally the bundle can be written as a direct product $U_{(a)} \times \mathfrak{F}$ where $U_{(a)}$ is a coordinate patch and the fibre $\mathfrak{F}$ any topological space. A structure group H can act on the fibre $\mathfrak{F}$. A section x is a generalization of a function on a manifold, in the sense that on each coordinate patch $U_{(a)}$ there is a function $x_{(a)} : U_{(a)} \longrightarrow \mathfrak{F}$. On a non-void intersection $U_{(a)} \cap U_{(b)}$ the section x in local coordinates satisfies $x_{(b)} = g_{(ba)} x_{(a)}$, where $g_{(ba)}$ is an element of the structure group H called transition function or cocycle. These transition functions satisfy the following three equations

$$g_{(aa)} = \mathbb{1}, \quad g_{(ab)} g_{(ba)} = \mathbb{1} \quad \text{and} \quad g_{(ab)} g_{(bc)} g_{(ca)} = \mathbb{1}, \tag{B.1}$$

on the corresponding non-empty overlaps of $U_{(a)}, U_{(b)}$ and $U_{(c)}$. The third one is called the cocycle condition.

We can now give a global definition of σ-model and matter scalars as promised in subsection 3.2. The σ-model scalars are the local coordinates of the Kähler manifold $\mathfrak{M}_\sigma$ and the matter scalars are sections of bundles over the manifold $\mathfrak{M}_\sigma$.

In subsection 3.3 matter representations were defined using their transformation properties under infinitesimal isometries. The task here is to extend the definitions of these matter representations over the whole manifold. To do this, these matter representations have to be interpreted as sections of bundles. The scalar field x^α transformed under infinitesimal isometries of $\mathfrak{M}_\sigma$ according to eq. (3.10).

We now turn to a few examples of bundles over Kähler manifolds. Let x^α be the components of a tangent vector in $T_p\mathfrak{M}_\sigma$ for each point $p \in \mathfrak{M}_\sigma$. The disjoint union of tangent spaces defines the tangent bundle $T\mathfrak{M}_\sigma$. A section x of the tangent bundle transforms under a change of coordinates on the intersection of two patches as

$$x^\beta_{(b)} = \big(X_{(ba)}\big)^\beta{}_\alpha \, x^\alpha_{(a)}, \tag{B.2}$$

with the matrix $X_{(ba)}$ defined in eq. (A.5). The matrices $X_{(ba)}$ satisfy all properties (B.1) of the transition functions of bundles, using the chain rule of differentiation. Sections of the cotangent bundle $T^*\mathfrak{M}_\sigma$ can be defined in a similar fashion. By taking sections of tensor products of the tangent and cotangent bundles, the tensor matter representation (3.12) can be defined over the whole manifold.

The next matter representation discussed in subsection 3.3 was a single chiral multiplet that transforms non-trivially under the action of the Killing vectors (3.14). The scalar s of the supermultiplet Ω is a section of a complex line bundle. A section of a complex line bundle is introduced by defining its transformation property when changing coordinate patches

$$s_{(b)} \longrightarrow s_{(a)} = e^{-q\,f^2 F_{(ab)}} s_{(b)}. \tag{B.3}$$

A particularly important example is the volume bundle with the section v that transforms under transition of coordinate patches as

$$v_{(b)} = \det X_{(ba)} v_{(a)} = e^{\mathrm{tr}\ln X_{(ba)}} v_{(a)}, \tag{B.4}$$

as it is always easy to obtain this bundle. In order that the transition function of the transformation (B.3) satisfies the cocycle conditions (B.1) we must have that

$$F_{(aa)} = 0, \quad F_{(ab)} + F_{(ba)} = 0, \quad F_{(abc)} \equiv F_{(ab)} + F_{(bc)} + F_{(ca)} \in \frac{2\pi i}{qf^2}\mathbb{Z}. \tag{B.5}$$

Because of the transformation rules (B.3) and (3.9) when changing between coordinate patches, the Kähler potential for s defined by

$$K_{line} = e^{qf^2 K_\sigma}\bar{s}s, \tag{B.6}$$

can be extended over the whole Kähler manifold $\mathfrak{M}_\sigma$. If we restrict ourselves to the bosonic part of the D-term of the superfield described by K_{line} in the globally supersymmetric case we obtain

$$\big[K_{line}\big]_D = e^{qf^2 K_\sigma} \left|D_\mu s\right|^2 + \ldots\,, \tag{B.7}$$

with the covariant derivate

$$D_\mu s = (\partial_\mu + B_\mu)\, s. \tag{B.8}$$

and the pullback connection $B_\mu = B_\alpha \partial_\mu z^\alpha$ given by

$$B = B_\alpha dz^\alpha \wedge = B_\mu dx^\mu \wedge = f^2 \boldsymbol{\partial} K_\sigma, \tag{B.9}$$

using the exterior derivative $\boldsymbol{\partial}$. Their conjugates $\bar{B}_\mu$ and $\bar{B}_{\underline{\alpha}}$ are defined in a similar fashion. This connection transforms as

$$B_\alpha^{(b)} \longrightarrow B_\alpha^{(a)} = B_\alpha^{(b)} + f^2 \partial_\alpha F_{(ab)} \tag{B.10}$$

under the change of coordinate patches, so that the derivative (B.8) is covariant.

We now turn to a crucial result relating the integral over a 2-cycle of the Kähler form to the cocycle condition of the line bundle. First note that

$$\omega(K_\sigma) = -i\mathbf{d}\boldsymbol{\partial} K_\sigma = i\mathbf{d}\bar{\boldsymbol{\partial}} K_\sigma = \frac{1}{f^2}\mathbf{d}\mathrm{Im}\, B \tag{B.11}$$

using definition (A.16). The analyticity of $F_{(ab)}$ implies that the transformation rules for B and $\bar{B}$ can be combined to

$$\mathrm{Im}\, B_{(b)} = \mathrm{Im}\, B_{(a)} + f^2 \mathbf{d}\mathrm{Im}\, F_{(ab)}. \tag{B.12}$$

Consider a 2-cycle C_2 that is covered with coordinate patches such that there is exactly one triple overlap region Δ. The integral of the Kähler form $\omega(K_\sigma)$ over the 2-cycle is computed as follows

$$\int_{C_2} \omega(K_\sigma) = \int_\Delta \omega(K_\sigma) = \frac{1}{f^2}\int_{\partial\Delta} \mathrm{Im}\, B = \sum_{(ab)} \mathrm{Im}\, F_{(ab)} = \mathrm{Im}\, F_{(abc)}. \tag{B.13}$$

Here we have used that the integral over C_2 can be split into integrals over subsets of C_2 contained in one coordinate patch only except for the integral over Δ. We have sketched the situation in figure B.1. Using Stoke's theorem all the integrals except those over Δ vanish. The integral over Δ is turned into an integral over the boundary $\partial\Delta$ using Stoke's again. As $\mathrm{Im}\, B$ is not a function on the manifold, we have to take (B.12) into account. On the other hand we have that the consistency of the line bundle requires eq. (B.5) to hold, hence the integral of the Kähler form over any 2-cycle satisfies

$$\frac{qf^2}{2\pi}\int_{C_2} \omega(K_\sigma) \in \mathbb{Z}. \tag{B.14}$$

Kähler forms that satisfy this condition are called Hodge and the Kähler manifolds having a Hodge Kähler form, are called Kähler-Hodge manifolds.

A useful formula to calculate the cohomology groups is the Künneth formula. Let $\mathfrak{M}$ be a product of two manifolds $\mathfrak{M} = \mathfrak{M}_1 \times \mathfrak{M}_2$. The rth cohomology group can be expressed in terms of the cohomology groups of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ as

$$H^r(\mathfrak{M}) = \bigoplus_{p+q=r} H^p(\mathfrak{M}_1) \otimes H^q(\mathfrak{M}_2). \tag{B.15}$$

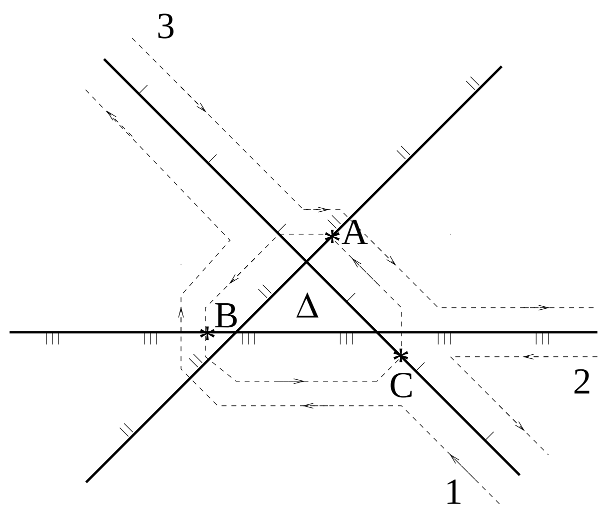

Figure B.1: The integral over a 2-cycle can be turned into an integral over the triple overlap. Here we just depict the part of the 2-cycle where there is a triple overlap; the boundaries of three different coordinate patches is indicated by the lines with I, II and III on them. The dotted lines give the orientation of boundaries of the regions over which we integrate. The integral over regions 1, 2 and 3 are zero using Stoke's theorem, because these regions are contractible. On the other hand the integral around the triple overlap Δ may be non-vanishing because the integral over the boundary of Δ goes through different coordinate patches.

Appendix C

Non-Holomorphic Transformations

This appendix discusses non-holomorphic transformations which are used in section 3.6. Since a Kähler manifold is complex, the coordinate transformations preserving the complex structure are holomorphic

$$Z^{\mathcal{A}} \longrightarrow Z'^{\mathcal{A}'} = \mathcal{R}^{\mathcal{A}'}(Z), \qquad \bar{Z}^{\underline{\mathcal{A}}} \longrightarrow \bar{Z}'^{\underline{\mathcal{A}}'} = \bar{\mathcal{R}}^{\underline{\mathcal{A}}'}(\bar{Z}). \tag{C.1}$$

Any object $V^{\mathcal{A}}$ and its conjugate $\bar{V}^{\underline{\mathcal{A}}}$ transforming as

$$V^{\mathcal{A}} \longrightarrow V'^{\mathcal{A}'} = X^{\mathcal{A}'}_{\mathcal{A}}(Z)V^{\mathcal{A}}, \qquad \bar{V}^{\underline{\mathcal{A}}} \longrightarrow \bar{V}'^{\underline{\mathcal{A}}'} = \bar{X}^{\underline{\mathcal{A}}'}_{\underline{\mathcal{A}}}(\bar{Z})\bar{V}^{\underline{\mathcal{A}}} \tag{C.2}$$

under the holomorphic coordinate transformations, with

$$X^{\mathcal{A}'}_{\mathcal{A}}(Z) = \mathcal{R}^{\mathcal{A}'}_{,\mathcal{A}}(Z), \quad \bar{X}^{\underline{\mathcal{A}}'}_{\underline{\mathcal{A}}}(Z) = \bar{\mathcal{R}}^{\underline{\mathcal{A}}'}_{,\underline{\mathcal{A}}}(Z), \tag{C.3}$$

is a covariant vector of the Kähler manifold. In the context of supersymmetric σ-models many covariant vectors are encountered, to name a few: the derivatives $\partial_\mu Z^{\mathcal{A}}$, the differentials $dZ^{\mathcal{A}}$ and the superpartners $\psi^{\mathcal{A}}_L$ of $Z^{\mathcal{A}}$.

Instead of the holomorphic transformation rules (C.2) we can consider more general transformations

$$V^{\mathcal{A}} \longrightarrow V'^{\mathcal{A}'} = X^{\mathcal{A}'}_{\mathcal{A}}(\bar{Z}, Z)V^{\mathcal{A}}, \tag{C.4}$$

where $X^{\mathcal{A}'}_{\mathcal{A}}(\bar{Z}, Z)$ are possibly non-holomorphic functions. This type of transformations can be used to make the physical content of a field theory more transparent, as is illustrated in section 3.6. The first thing to note is that these transformations cannot be generated by non-holomorphic coordinate transformations, because they would introduce terms involving $\bar{V}^{\underline{\mathcal{A}}}$ in eq. (C.4). Therefore the transformations (C.4) can only be defined on the level of covariant vectors and geometrical objects like the metric: $V'^{\mathcal{A}'}$ is nothing but a short-hand notation for the expression $X^{\mathcal{A}'}_{\mathcal{A}}(\bar{Z}, Z)V^{\mathcal{A}}$ with a covariant vector $V^{\mathcal{A}}$. In the following we study how the transformations (C.4) change the appearance of formulae involving the metric, connection and curvature.

If we demand that the metric defines an invariant inner product for covariant vectors, it must transform as

$$G_{\underline{\mathcal{A}}\mathcal{A}} \longrightarrow G'_{\underline{\mathcal{A}}'\mathcal{A}'} = \bar{X}^{\underline{\mathcal{A}}}_{\underline{\mathcal{A}}'} X^{\mathcal{A}}_{\mathcal{A}'} G_{\underline{\mathcal{A}}\mathcal{A}} \tag{C.5}$$

where $X^{\mathcal{A}}_{\mathcal{A}'}(\bar{Z}, Z)$ is the inverse of $X^{\mathcal{A}'}_{\mathcal{A}}(\bar{Z}, Z)$.

A word about our notation is in order here: let $A_{\mathcal{A}}$ be any object with one index down, not necessarily a vector; it may be a function of covariant vectors and derivatives. Applying (C.4) to all covariant vectors transforms $A_{\mathcal{A}}$ into $A'_{\mathcal{A}'}$. One can also just contract $A_{\mathcal{A}}$ with the transformation matrix $X^{\mathcal{A}}_{\mathcal{A}'}$; this is denoted by $A_{\mathcal{A}'} = X^{\mathcal{A}}_{\mathcal{A}'} A_{\mathcal{A}}$. In the case of covariant vectors and the metric $G'_{\underline{\mathcal{A}}'\mathcal{A}'} = G_{\underline{\mathcal{A}}'\mathcal{A}'}$ these two definitions coincide but this is not true in general. (When there is no confusion possible, like with covariant vectors or the metric, we drop the prime on the symbol itself.) The prime example where there is a difference, is the connection

$$\begin{aligned}
\Gamma^{\mathcal{A}}_{\mathcal{B}\mathcal{C}} \longrightarrow \Gamma'^{\mathcal{A}'}_{\mathcal{B}'\mathcal{C}'} &= \Gamma^{\mathcal{A}'}_{\mathcal{B}'\mathcal{C}'} + U^{\mathcal{A}'}_{\mathcal{B}'\mathcal{C}'} + G^{\mathcal{A}'\underline{\mathcal{B}}'} \bar{U}^{\underline{\mathcal{A}}'}_{\underline{\mathcal{B}}'\mathcal{C}'} G_{\underline{\mathcal{A}}'\mathcal{B}'} \\
\bar{\Gamma}^{\underline{\mathcal{A}}}_{\underline{\mathcal{B}}\underline{\mathcal{C}}} \longrightarrow \bar{\Gamma}'^{\underline{\mathcal{A}}'}_{\underline{\mathcal{B}}'\underline{\mathcal{C}}'} &= \bar{\Gamma}^{\underline{\mathcal{A}}'}_{\underline{\mathcal{B}}'\underline{\mathcal{C}}'} + \bar{U}^{\underline{\mathcal{A}}'}_{\underline{\mathcal{B}}'\underline{\mathcal{C}}'} + G^{\mathcal{B}'\underline{\mathcal{A}}'} U^{\mathcal{A}'}_{\mathcal{B}'\underline{\mathcal{C}}'} G_{\underline{\mathcal{B}}'\mathcal{A}'}
\end{aligned} \tag{C.6}$$

with

$$\begin{aligned}
U^{\mathcal{A}'}_{\mathcal{B}'\mathcal{C}'} &= X^{\mathcal{A}'}_{\mathcal{A}} X^{\mathcal{A}}_{\mathcal{B}',\mathcal{C}} X^{\mathcal{C}}_{\mathcal{C}'}, \qquad & \bar{U}^{\underline{\mathcal{A}}'}_{\underline{\mathcal{B}}'\underline{\mathcal{C}}'} &= \bar{X}^{\underline{\mathcal{A}}'}_{\underline{\mathcal{A}}} X^{\underline{\mathcal{A}}}_{\underline{\mathcal{B}}',\underline{\mathcal{C}}} X^{\underline{\mathcal{C}}}_{\underline{\mathcal{C}}'}, \\
U^{\mathcal{A}'}_{\mathcal{B}'\underline{\mathcal{C}}'} &= X^{\mathcal{A}'}_{\mathcal{A}} X^{\mathcal{A}}_{\mathcal{B}',\underline{\mathcal{C}}} X^{\underline{\mathcal{C}}}_{\underline{\mathcal{C}}'}, \qquad & \bar{U}^{\underline{\mathcal{A}}'}_{\underline{\mathcal{B}}'\mathcal{C}'} &= \bar{X}^{\underline{\mathcal{A}}'}_{\underline{\mathcal{A}}} \bar{X}^{\underline{\mathcal{A}}}_{\underline{\mathcal{B}}',\mathcal{C}} X^{\mathcal{C}}_{\mathcal{C}'}.
\end{aligned} \tag{C.7}$$

Notice that the third term in equations (C.6) vanishes if the transformations are holomorphic. Here we see clearly that the connection is not a tensor even in the case of holomorphic transformations. But this exactly enables us to define a covariant derivative $\mathcal{D}_\mu$ for covariant vectors $\mathcal{D}_\mu V^{\mathcal{A}} \equiv \partial_\mu V^{\mathcal{A}} + \Gamma^{\mathcal{A}}_{\mathcal{C}\mathcal{B}} \partial_\mu Z^{\mathcal{B}} V^{\mathcal{C}}$. However, it is only covariant under holomorphic transformations and not under eq.(C.4); indeed

$$(\mathcal{D}_\mu V)'^{\mathcal{A}'} = \mathcal{D}_\mu V^{\mathcal{A}'} + G^{\mathcal{A}'\underline{\mathcal{B}}'} \bar{U}^{\underline{\mathcal{A}}'}_{\underline{\mathcal{B}}'\mathcal{B}'} G_{\underline{\mathcal{A}}'\mathcal{C}'} \partial_\mu Z^{\mathcal{B}'} V^{\mathcal{C}'} - U^{\mathcal{A}'}_{\mathcal{C}'\underline{\mathcal{B}}'} \partial_\mu \bar{Z}^{\underline{\mathcal{B}}'} V^{\mathcal{C}'}. \tag{C.8}$$

The second term on the r.h.s. follows from eq. (C.6) and the third arises, because the ordinary derivative ∂_μ within $\mathcal{D}_\mu$ can hit the transformation matrix $X^{\mathcal{A}'}_{\mathcal{A}}$ which may also depend on $\bar{Z}^{\underline{\mathcal{A}}}$. The first term on the r.h.s. is of the same form one would get if the transformations (C.4) were holomorphic. The last two terms involve U and $\bar{U}$'s with mixed indices indicating the non-holomorphic nature of (C.4).

Finally we investigate how the transformations (C.4) influence the curvature. The calculation follows the same line as above, but now it is really convenient to separate terms which do not have mixed transformations involving U and $\bar{U}$. With this separation one can identify which terms behave as if the transformations (C.4) are holomorphic. We call these terms holomorphic and indicate them with a superscript H. The remaining terms have U's and $\bar{U}$'s with mixed indices. They are called non-holomorphic and are indicated by a superscript N.

As the curvature is a tensor under holomorphic transformations, the holomorphic part R^H also transforms as a tensor under (C.4). By identifying the holomorphic and non-holomorphic parts we find

$$\begin{aligned} R'{}_{\underline{\mathcal{A}}'\mathcal{A}'\underline{\mathcal{B}}'\mathcal{B}'} = \quad & R^H{}_{\underline{\mathcal{A}}'\mathcal{A}'\underline{\mathcal{B}}'\mathcal{B}'} + G^N{}_{\underline{\mathcal{A}}'\mathcal{A}',\underline{\mathcal{B}}'\mathcal{B}'} - G^H{}_{\underline{\mathcal{A}}'\mathcal{C}',\underline{\mathcal{B}}'} G^{\mathcal{C}'\underline{\mathcal{C}}'} G^N{}_{\underline{\mathcal{C}}'\mathcal{A}',\mathcal{B}'} + \\ & - G^N{}_{\underline{\mathcal{A}}'\mathcal{C}',\underline{\mathcal{B}}'} G^{\mathcal{C}'\underline{\mathcal{C}}'} G^H{}_{\underline{\mathcal{C}}'\mathcal{A}',\mathcal{B}'} + G^N{}_{\underline{\mathcal{A}}'\mathcal{C}',\underline{\mathcal{B}}'} G^{\mathcal{C}'\underline{\mathcal{C}}'} G^N{}_{\underline{\mathcal{C}}'\mathcal{A}',\mathcal{B}'}. \end{aligned} \tag{C.9}$$

As we already know how the holomorphic part of the curvature transforms, we only consider the terms with non-holomorphic transformations. In these terms, we replace the remaining holomorphic parts like $G^H{}_{\underline{\mathcal{A}}'\mathcal{C}',\mathcal{B}'}$ by $(G' - G^N)_{\underline{\mathcal{A}}'\mathcal{C}',\mathcal{B}'}$. In applications we are not so much interested in the transformed curvature itself, but more in having a simple way to write expressions involving the curvature, like the four-fermion terms. Therefore we write

$$R'{}_{\underline{\mathcal{A}}'\mathcal{A}'\underline{\mathcal{B}}'\mathcal{B}'} = \hat{R}'_{\underline{\mathcal{A}}'\mathcal{A}'\underline{\mathcal{B}}'\mathcal{B}'} + G'{}_{\underline{\mathcal{A}}'\mathcal{A}',\underline{\mathcal{C}}'} \bar{U}^{\underline{\mathcal{C}}'}_{\underline{\mathcal{B}}'\mathcal{B}'}. \tag{C.10}$$

Notice that the second term depends on the order of the indices $\underline{\mathcal{B}}'$ and $\mathcal{B}'$. The first term is given by

$$\begin{aligned} \hat{R}'_{\underline{\mathcal{A}}'\mathcal{A}'\underline{\mathcal{B}}'\mathcal{B}'} = & R_{\underline{\mathcal{A}}'\mathcal{A}'\underline{\mathcal{B}}'\mathcal{B}'} + G_{\underline{\mathcal{D}}'\mathcal{A}'} \bar{W}^{\underline{\mathcal{D}}'}_{\underline{\mathcal{A}}'\underline{\mathcal{B}}'\mathcal{B}'} + G_{\underline{\mathcal{A}}'\mathcal{D}'} W^{\mathcal{D}'}_{\mathcal{A}'\underline{\mathcal{B}}'\mathcal{B}'} \\ & + G_{\underline{\mathcal{A}}'\mathcal{D}'} U^{\mathcal{D}'}_{\mathcal{C}'\underline{\mathcal{B}}'} G^{\mathcal{C}'\underline{\mathcal{C}}'} \bar{U}^{\underline{\mathcal{D}}'}_{\underline{\mathcal{C}}'\mathcal{B}'} G_{\underline{\mathcal{D}}'\mathcal{A}'} - \bar{U}^{\underline{\mathcal{D}}'}_{\underline{\mathcal{A}}'\mathcal{B}'} G_{\underline{\mathcal{D}}'\mathcal{D}'} U^{\mathcal{D}'}_{\mathcal{A}'\underline{\mathcal{B}}'} \\ & - G_{\underline{\mathcal{A}}'\mathcal{D}'} \left(-\Gamma'^{\mathcal{D}'}_{\mathcal{E}'\mathcal{B}'} U^{\mathcal{E}'}_{\mathcal{A}'\underline{\mathcal{B}}'} + U^{\mathcal{D}'}_{\mathcal{E}'\underline{\mathcal{B}}'} \Gamma'^{\mathcal{E}'}_{\mathcal{A}'\mathcal{B}'} + U^{\mathcal{D}'}_{\mathcal{E}'\mathcal{B}'} U^{\mathcal{E}'}_{\mathcal{A}'\underline{\mathcal{B}}'} \right) \\ & - G_{\underline{\mathcal{D}}'\mathcal{A}'} \left(-\bar{\Gamma}'^{\underline{\mathcal{D}}'}_{\underline{\mathcal{E}}'\underline{\mathcal{B}}'} \bar{U}^{\underline{\mathcal{E}}'}_{\underline{\mathcal{A}}'\mathcal{B}'} + \bar{U}^{\underline{\mathcal{D}}'}_{\underline{\mathcal{E}}'\mathcal{B}'} \bar{\Gamma}'^{\underline{\mathcal{E}}'}_{\underline{\mathcal{A}}'\underline{\mathcal{B}}'} + \bar{U}^{\underline{\mathcal{D}}'}_{\underline{\mathcal{E}}'\underline{\mathcal{B}}'} \bar{U}^{\underline{\mathcal{E}}'}_{\underline{\mathcal{A}}'\mathcal{B}'} \right). \end{aligned} \tag{C.11}$$

Here W is defined as $W^{\mathcal{D}'}_{\mathcal{A}'\underline{\mathcal{B}}'\mathcal{B}'} = X^{\mathcal{D}'}_{\mathcal{D}} X^{\mathcal{D}}_{\mathcal{A}',\underline{\mathcal{B}}\mathcal{B}} \bar{X}^{\underline{\mathcal{B}}}_{\underline{\mathcal{B}}'} X^{\mathcal{B}}_{\mathcal{B}'}$ and a similar definition holds for $\bar{W}$.

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